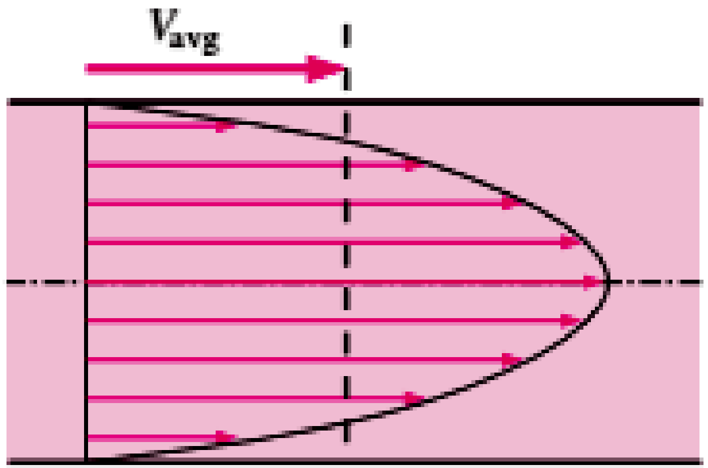
Real Fluid Flow

The difference between an ideal fluid and a real fluid is that in case of a real fluid, ***viscous effects take place in the vicinity of solid boundary***. As the ***ideal fluid is inviscid*** the presence of a solid boundary does not affect the flow, ***the fluid just slips over the boundary*** and the ***velocity distribution over the boundary*** is ***uniform throughout***. **For a real fluid** approaches a boundary surface, the fluid particles which come in contact with the stationary surface are brought to rest ***by viscous resistance*** of the fluid. So ***the fluid particles at the boundary have zero velocity*** relative to the boundary (no-slip condition). ***The fluid particles away from the boundary attain higher velocity***, and ***the change of velocity across the flow gives rise to*** a ***velocity gradient***. ***This velocity gradient is responsible for development of viscous shear resistance, which oppose the motion***.

In fluid flow, it is convenient to work with an ***average velocity*** V_{avg}, which remains constant in incompressible flow when the cross-sectional area of the pipe is constant.



Also, **the friction** between the fluid particles in a pipe does ***cause a slight rise in***

fluid temperature as a result of the mechanical energy being converted to sensible thermal energy. But this temperature rise due to ***frictional heating*** is usually too small to warrant any consideration in calculations and thus is disregarded. For example, in the absence of any heat transfer, no noticeable difference can be detected between the inlet and outlet temperatures of water flowing in a pipe. ***The primary consequence of friction in fluid flow is pressure drop, and thus any significant temperature change in the fluid is due to heat transfer.***

The ***conservation of mass***

$$\boxed{\dot{m} = \rho V_{avg} A_c = \int_{A_c} \rho u(r)\, dA_c}$$

where $m.$ is the mass flow rate, ρ is the density, A_c is the x-sectional area, and $u(r)$ is the velocity profile. Then ***the average velocity for incompressible flow in a circular pipe*** of radius R can be expressed as

$$\boxed{V_{avg} = \frac{\int_{A_c} \rho u(r)\, dA_c}{\rho A_c} = \frac{\int_0^R \rho u(r) 2\pi r\, dr}{\rho \pi R^2} = \frac{2}{R^2}\int_0^R u(r) r\, dr}$$

Therefore, ***when the flow rate or the velocity profile is known, the average velocity can be determined easily***.

Laminar or Turbulent Flow

The flow of a fluid in a pipe may be laminar flow or it may be turbulent flow. Osborne Reynolds (1842–1912), a British scientist and mathematician, was the first to distinguish the difference between these two classifications of flow by using a simple apparatus as shown by the figure in the margin, which is a ***sketch of Reynolds' dye experiment***. Reynolds injected dye into a pipe in which water flowed due to gravity. The entrance region of the pipe is depicted in Fig.*a*. ***The flow*** ceases being ***smooth and steady (laminar)*** and becomes ***fluctuating and agitated (turbulent)***. ***The changeover is called transition to turbulence.***

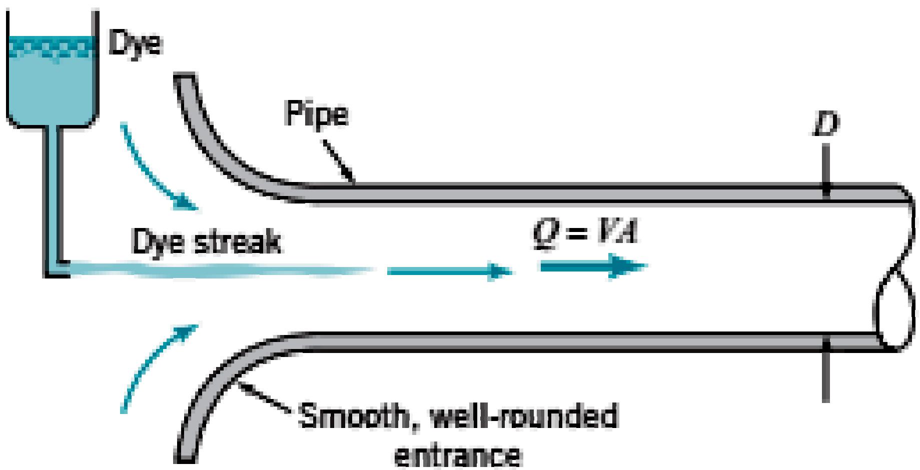
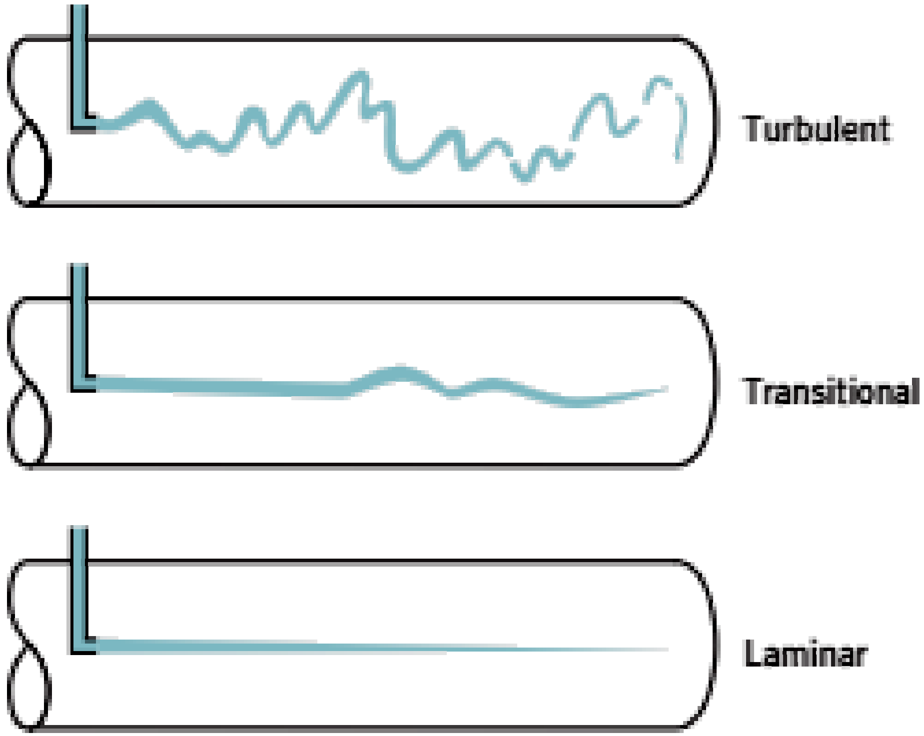


Fig. *a*



(*b*)

For "**small enough flowrates**" the ***dye streak*** (a streakline) will remain as a ***well-defined line as it flows along***, with ***only slight blurring due to molecular diffusion of the dye*** into the surrounding water. This characteristics is denoted as **laminar Flow**.

For a somewhat larger "**intermediate flowrate**" ***the dye streak fluctuates in time and space***, and ***intermittent bursts of irregular behavior appear along the streak***. This characteristics is denoted as **transitional flow**.

On the other hand, for "**large enough flowrates**" the ***dye streak*** almost immediately ***becomes blurred and spreads across the entire pipe in a random fashion***. This characteristics is denoted as **turbulent flow.**

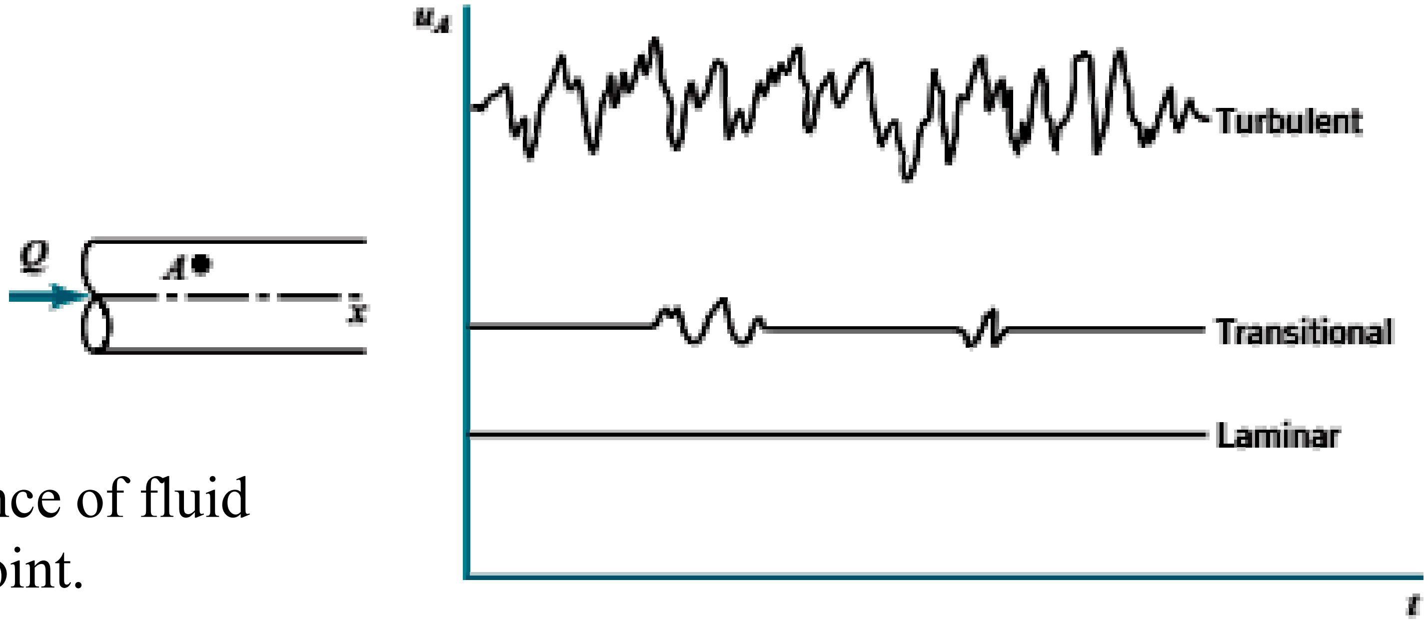


Fig: Time dependence of fluid velocity at a point.

Transition depends on many effects, such as ***geometry***, ***surface roughness***, ***flow velocity***, ***surface temperature***, ***and*** ***type of fluid***, among other things.

The flow is controlled by **(i)** pressure gradient **(ii)** the pipe diameter or hydraulic mean diameter **(iii)** the fluid properties like viscosity and density and **(iv)** the pipe roughness.

The flow regime in the first case is said to be **laminar**, characterized by ***smooth streamlines* and *highly ordered motion***, and **turbulent** in the second case, where it is characterized by ***velocity fluctuations* and *highly disordered motion***. The **transition** from laminar to turbulent flow does not occur suddenly; rather, it occurs over some region ***in which the flow fluctuates between laminar and turbulent flows*** before it becomes fully turbulent. ***The velocity at which the flow changes from the laminar to turbulent for the case of a given fluid at a given temperature and in a given pipe is*** known as **critical velocity**. Most flows encountered in practice are turbulent. ***Laminar flow is encountered when highly viscous fluids such as oils flow in small pipes or narrow passages such as blood flow.***

Reynolds Number

On the basis of his experiments Reynolds discovered that ***the occurrence of a laminar and turbulent flow was governed by the relative magnitudes of the inertia and the viscous forces***. It was indicated by Reynolds that at low velocities of flow, even for the fluids having very small viscosity, the viscous forces become predominant and therefore, the flow is largely viscous in character. However, at higher velocities of flow the inertial forces have predominance over the viscous forces. Reynolds related the inertia to viscous forces and arrived at a dimensionless parameter. ***The Reynolds number is the ratio of the inertial force to the viscous force.***

$$\boxed{\text{Re or } N_R = \frac{\text{Inertia force}}{\text{Viscous force}} = \frac{F_i}{F_v}}$$

According to **Newton's second law of motion the inertia force F_i**, is given by

$$F_i = \text{mass} \times \text{acceleration}$$
$$= \rho \times \text{volume} \times \text{acceleration}$$
$$= \rho \times L^3 \times (L/T^2) = (\rho L^2 V^2)$$

Similarly **viscous force F_v, is given by Newton's law of viscosity** as

$$F_v = \tau \times \text{area}$$

$$= \mu \frac{\partial v}{\partial y} \times L^2 = (\mu V L)$$

$$\boxed{\mathrm{Re} \text{ or } N_R = \frac{(\rho L^2 V^2)}{\mu V L} = \frac{\rho V L}{\mu}}$$

Where V_{avg} = average flow velocity (m/s), L = characteristic length of the geometry (diameter in case of pipe, in m), and $v = \mu/\rho$ kinematic viscosity of the fluid (m^2/s). Note that the Reynolds number is a *dimensionless* quantity. Also, **kinematic viscosity** can be viewed as ***viscous diffusivity*** or ***diffusivity for momentum***.

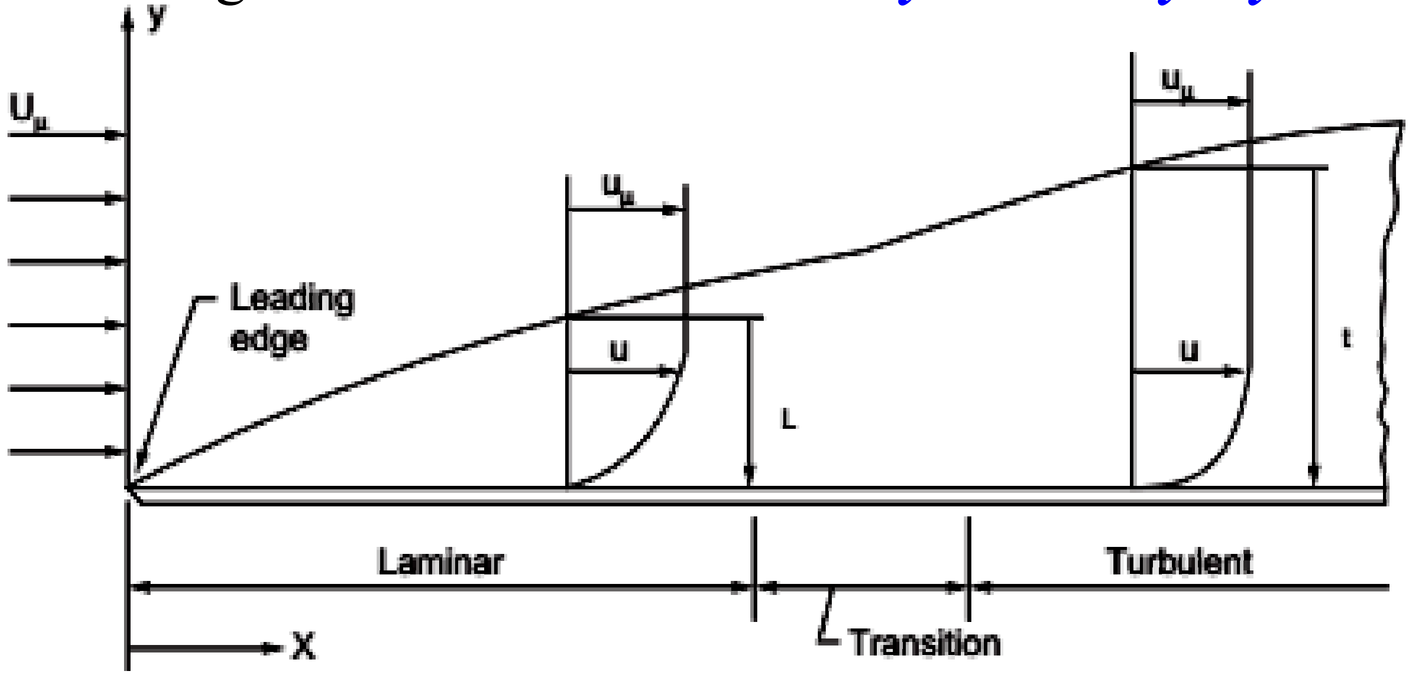
Viscous force ***tends to keep the layers moving smoothly one over the other***. **Inertia forces** ***tend to move the particles away from the layer***. ***When viscous force are sufficiently high so that any disturbance is smoothed down, laminar flow prevails in pipes***. **When velocity increases, inertia forces increase and particles are pushed upwards out of the smoother path.**

Under most practical conditions, **the flow in a circular pipe** is ***laminar for Re ≤ 2300, turbulent for Re ≥ 4000, and transitional in between***.

Boundary Layer

When fluids flow over surfaces, the molecules near the surface are brought to rest due to the viscosity of the fluid. The adjacent layers also slow down gradually as a result of friction, but to a lower and lower extent. This slowing down is found limited to a thin layer near the surface. ***The fluid beyond this layer is not affected by the presence of the surface***. ***To make up for this velocity reduction, the velocity of the fluid at the midsection of the pipe has to increase to keep the mass flow rate through the pipe constant***. As a result, a velocity gradient develops along the pipe. The fluid layer near the surface in which the effects of the viscous shearing forces caused by fluid viscosity are felt and there is a general slowing down is defined as ***velocity boundary layer***.

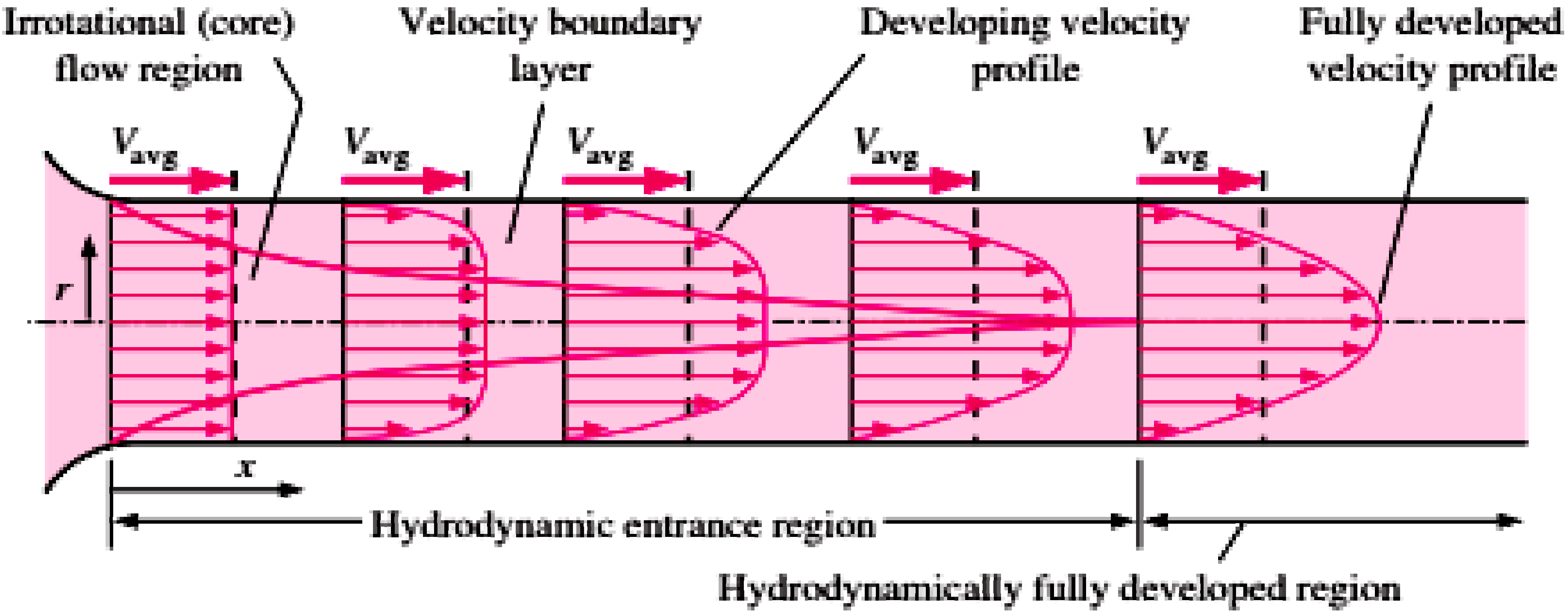
Fig: Boundary Layer Development (flat-plate)

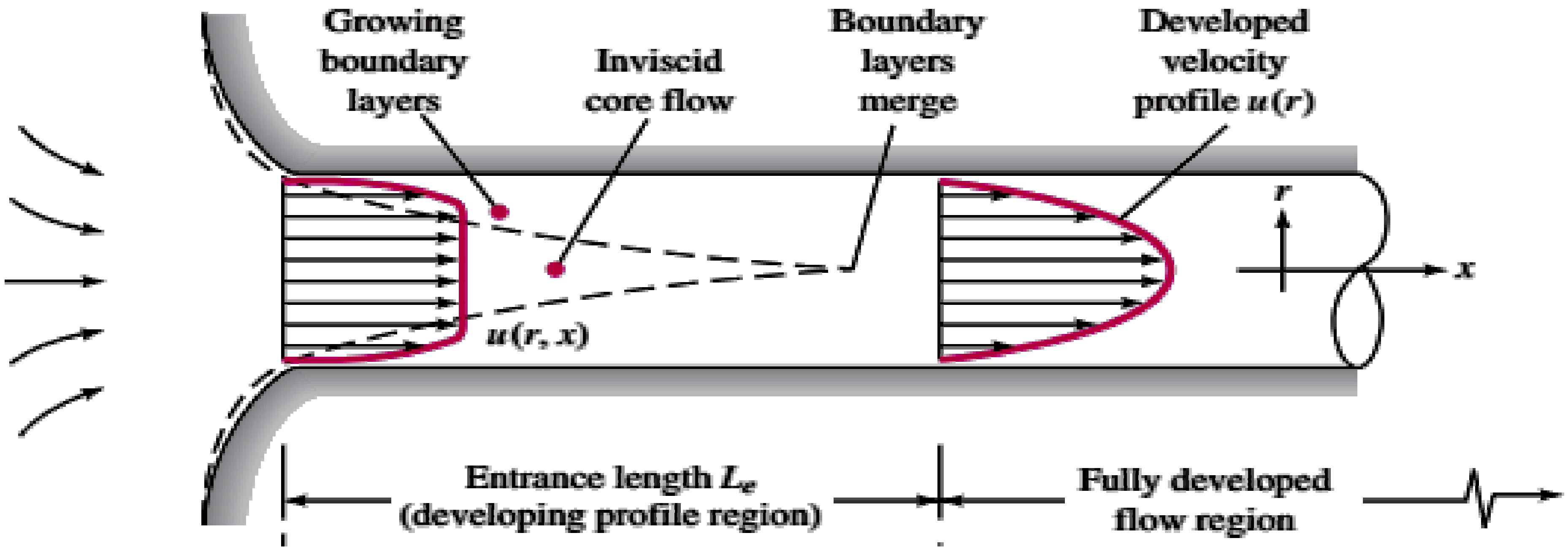


The velocity of flow in this layer increases from zero at the surface to free stream velocity at the edge of the boundary layer. The development of the boundary layer in flow over a flat plate and the velocity distribution in the layer are shown in above Fig. ***Pressure drop in fluid flow is to overcome the viscous shear force which depends on the velocity gradient at the surface***. Velocity gradient exists only in the boundary layer. The situation when a uniform flow meets with a plane surface parallel to the flow is shown in Fig. At the plane of entry (leading edge) the velocity is uniform and equals free stream velocity. Beyond this point, the fluid near the surface comes to rest and adjacent layers are retarded to a larger and larger depth as the flow proceeds.

The thickness of the boundary layer increases due to the continuous retardation of flow. The flow initially is laminar. There is no intermingling of layers. ***Momentum transfer is at the molecular level, mainly by diffusion***. The viscous forces predominate over inertia forces. Small disturbances are damped out. Beyond a certain distance, the flow in the boundary layer becomes turbulent with macroscopic mixing of layers. Inertia forces become predominant. This change occurs at a value of Reynolds number (given $Re = ux/v$, where v is the kinematic viscosity) of about **5×10^5** in the case of flow over flat plates.

Development Of Boundary Layer In Closed Conduits(Pipes)





In this case the boundary layer develops all over the circumference. The initial development of the boundary layer is similar to that over the flat plate. The thickness of this boundary layer increases in the flow direction until the boundary layer reaches the pipe center and thus fills the entire pipe, as shown in Fig. ***The boundary layers merge and further changes in velocity distribution becomes impossible***. ***The velocity profile beyond this point remains unchanged.***

The region from the pipe inlet to the point at which the boundary layer merges at the centerline is called the ***hydrodynamic entrance region***, and the length of this region is called the ***hydrodynamic entry length*** L_h. Flow in the entrance region is called ***hydrodynamically developing flow*** since this is the region where the velocity profile develops. The region beyond the entrance region in which the velocity profile is fully developed and remains unchanged is called the ***hydrodynamically fully developed region***. The flow is said to be **fully developed** when the normalized temperature profile remains unchanged as well.

Hydrodynamically developed flow is equivalent to fully developed flow when the fluid in the pipe is not heated or cooled since the fluid temperature in this case remains essentially constant throughout.

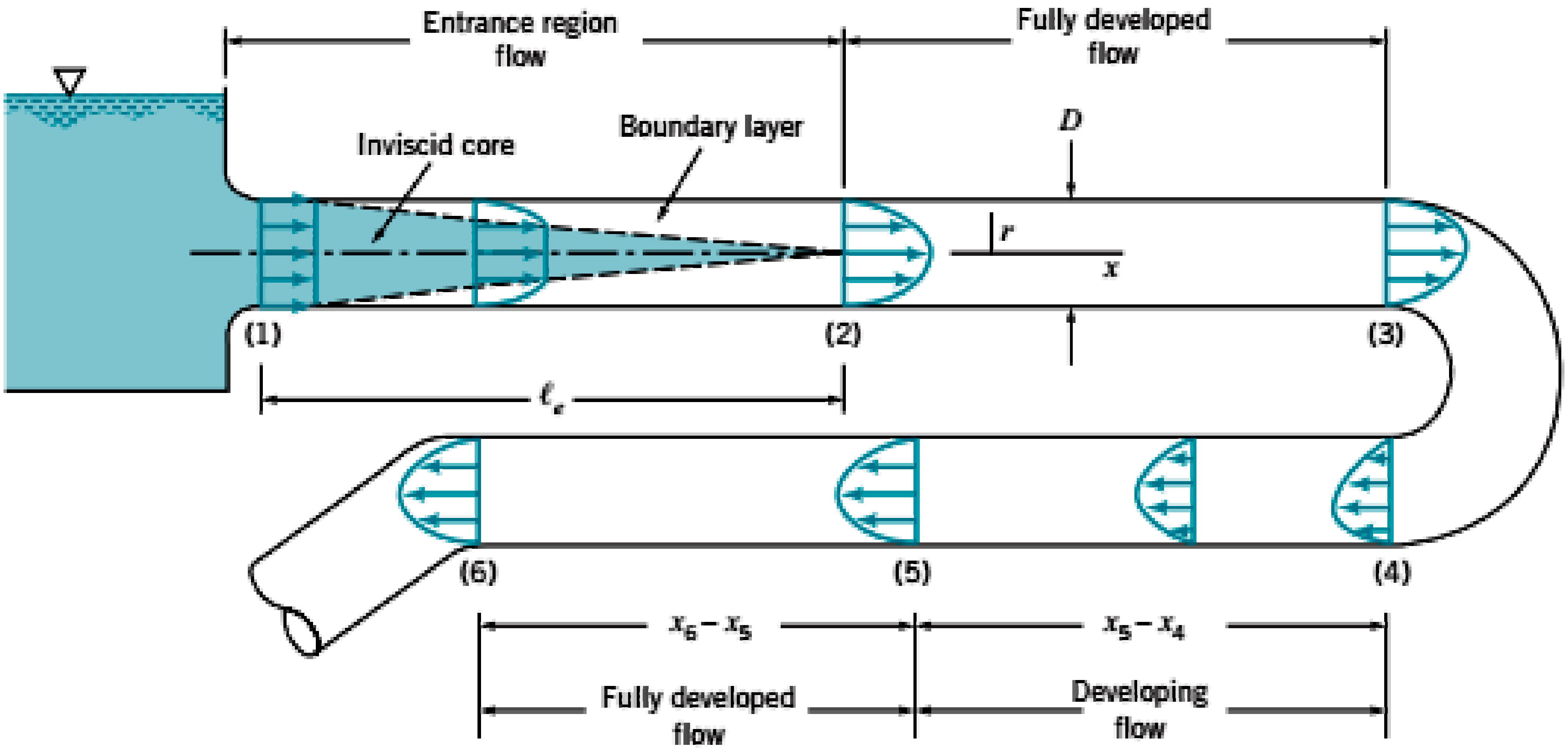


Fig: Entrance region, developing flow, and fully developed flow in a pipe system.

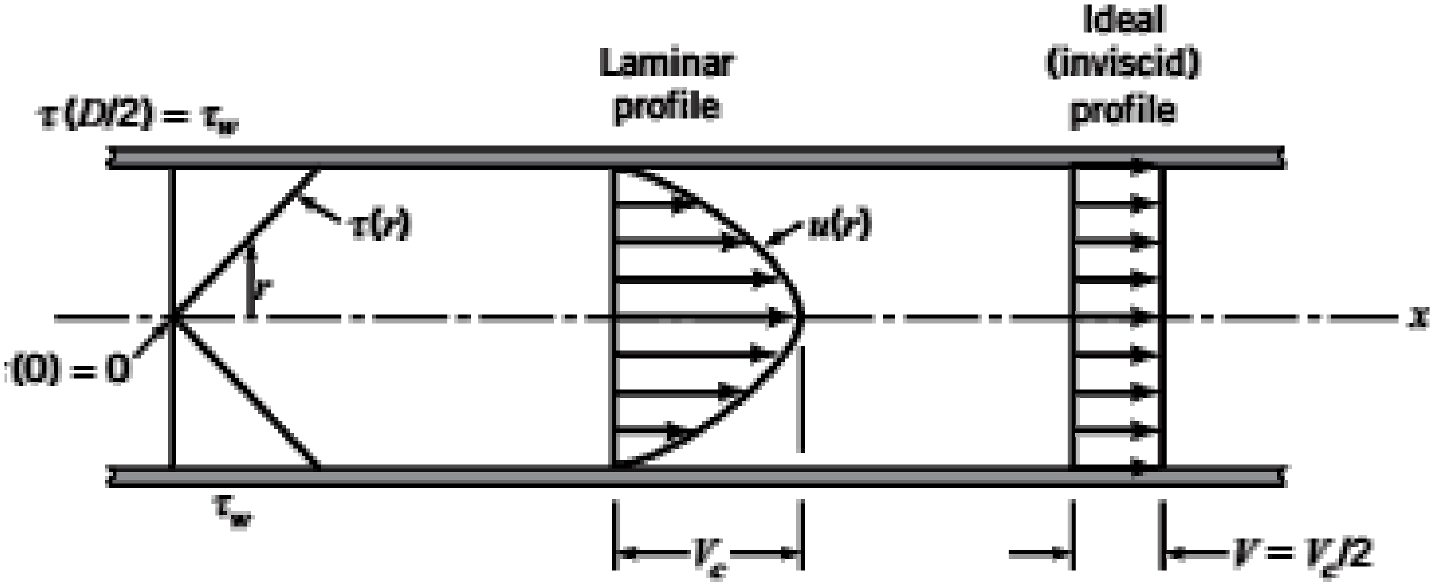


Fig: Shear stress distribution within the fluid in a pipe (laminar or turbulent flow) and typical velocity profiles.

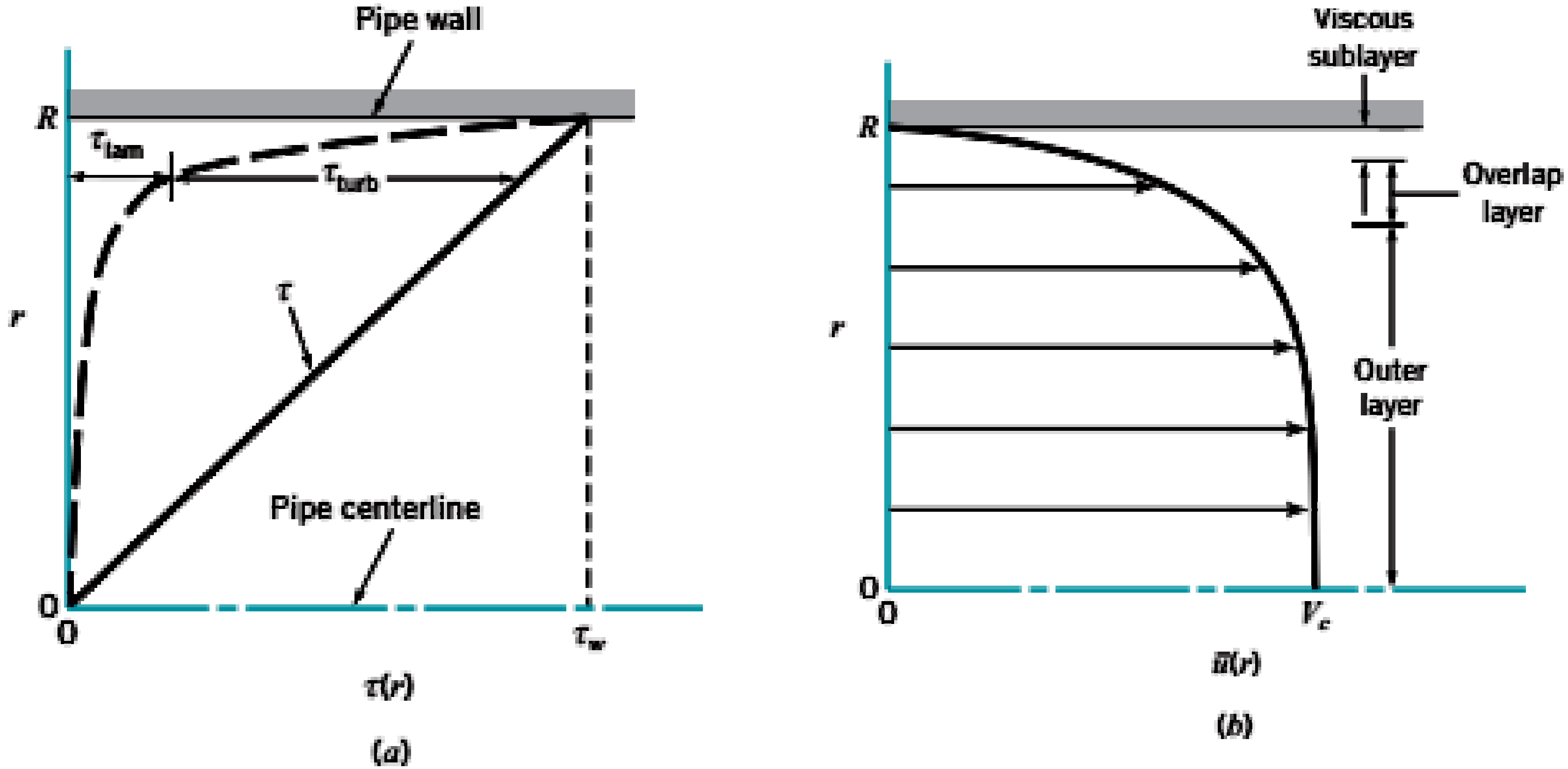


Fig: Structure of turbulent flow in a pipe. (*a*) Shear stress. (*b*) Average velocity

The velocity profile in the fully developed region is ***parabolic* in laminar flow** and somewhat ***flatter* (or *fuller*) in turbulent flow *due to eddy motion and more vigorous mixing in the radial direction***. **The time-averaged velocity profile remains unchanged when the flow is fully developed,** and thus

$$\boxed{\textit{Hydrodynamically fully developed:} \qquad \frac{\partial u(r,x)}{\partial x} = 0 \quad \rightarrow \quad u = u(r)}$$

The shear stress at the pipe wall τ_w is related to the slope of the velocity profile at the surface. Noting that the velocity profile remains unchanged ***in the hydrodynamically fully developed region, the wall shear stress also remains constant in that region.***

Consider fluid flow in the hydrodynamic entrance region of a pipe. ***The wall shear stress is the highest at the pipe inlet where the thickness of the boundary layer is smallest, and decreases gradually to the fully developed value***, as shown in following Fig. Therefore, ***the pressure drop is higher in the entrance regions of a pipe***, and ***the effect of the entrance region is always to increase the average friction factor for the entire pipe***. This increase may be significant for short pipes but is negligible for long ones.

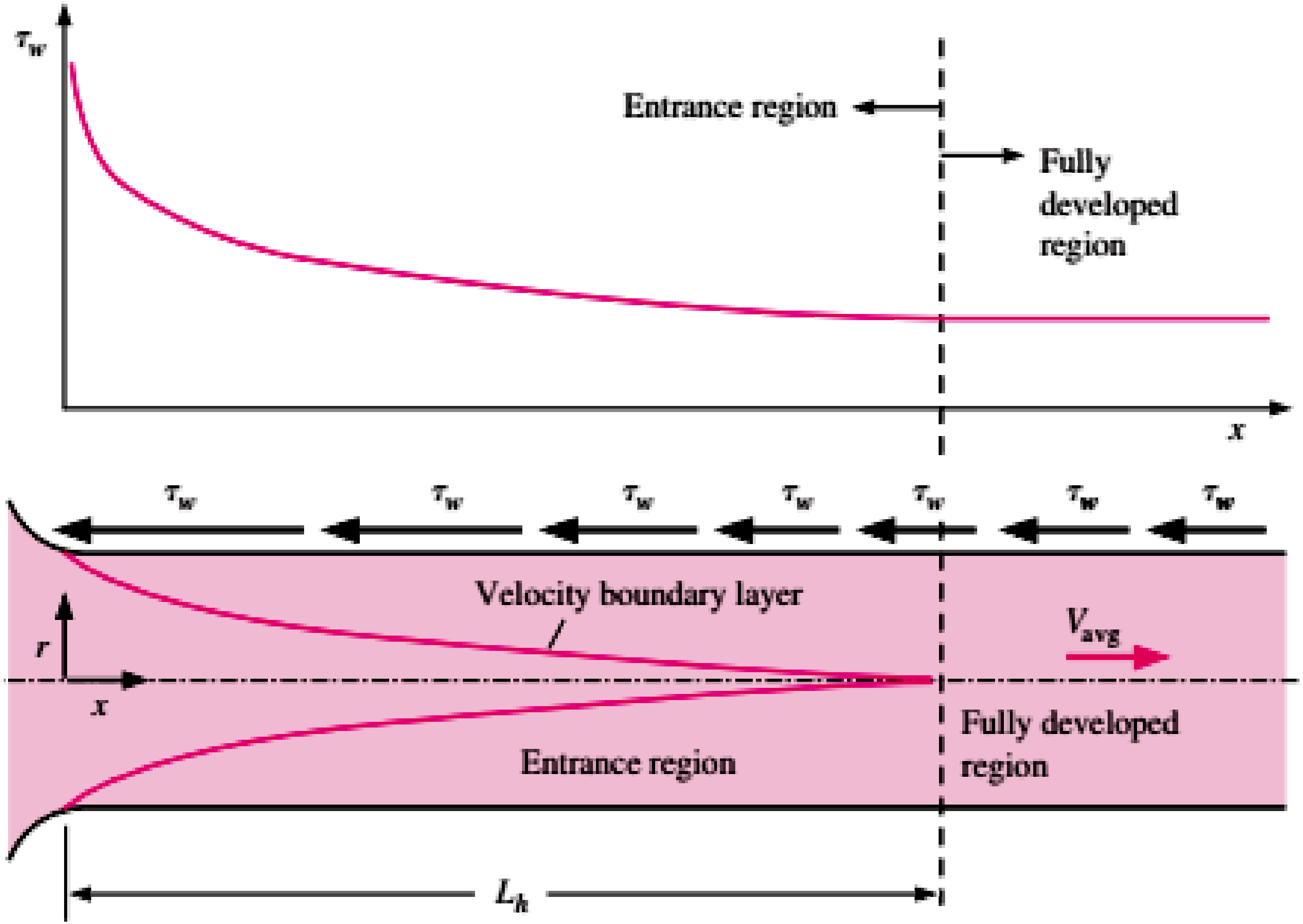
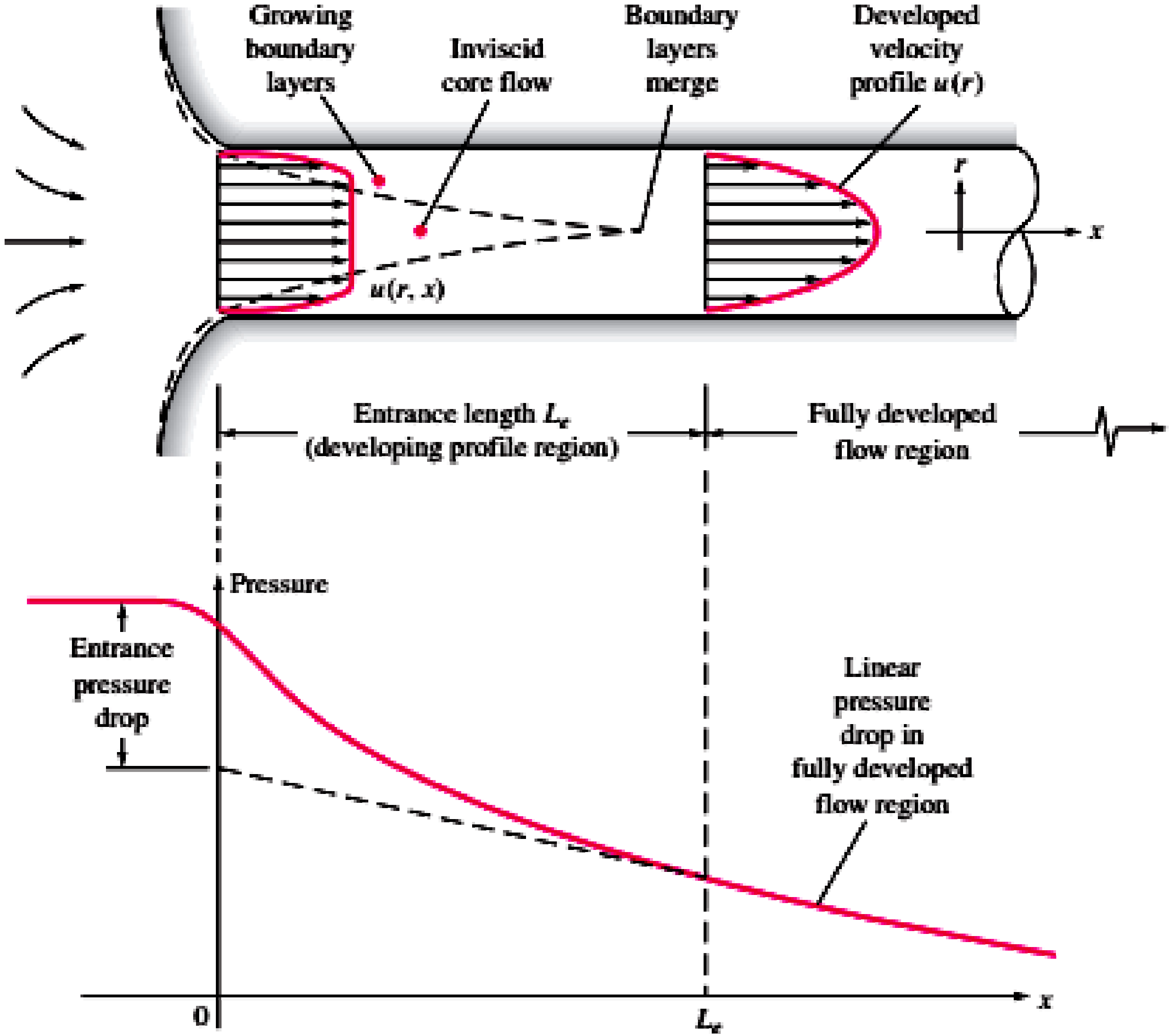


Fig: The variation of wall shear stress in the flow direction of flow in a pipe from the entrance region into the fully developed region.

Pressure and Shear Stress

Fully developed steady flow in a constant diameter pipe may be driven by gravity and/or pressure forces. **For horizontal pipe flow**, ***gravity has no effect except for a hydrostatic pressure variation across the pipe, that is usually negligible***. It is the pressure difference, $\Delta p = p_2 - p_1$ between two sections of the horizontal pipe which forces the fluid through the pipe. ***Viscous effects provide the restraining force that exactly balances the pressure force, thereby allowing the fluid to flow through the pipe with no acceleration***. ***If viscous effects were absent in such flows, the pressure would be constant throughout the pipe, except for the hydrostatic variation***.

In non-fully developed flow regions, such as the **entrance region** of a pipe, ***the fluid accelerates or decelerates*** as it flows (the velocity profile changes from a uniform profile at the entrance of the pipe to its fully developed profile at the end of the entrance region). Thus, ***in the entrance region there is a balance between pressure, viscous, and inertia (acceleration) forces***. ***The result is a pressure distribution along the horizontal pipe*** as shown in following Fig. ***The magnitude of the pressure gradient***, $\partial p/\partial x$ ***is larger in the entrance region than in the fully developed region, where it is a constant***, $\partial p/\partial x = -\Delta p/l < 0$.

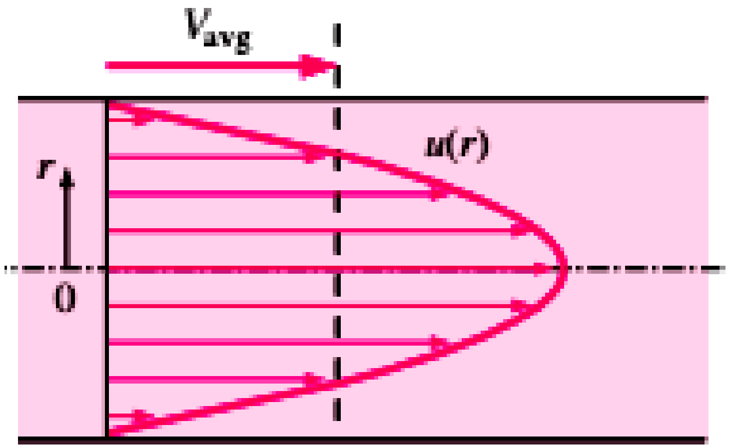


Growing boundary layers
Inviscid core flow
Boundary layers merge
Developed velocity profile $u(r)$
r
x
$u(r, x)$
Entrance length L_e (developing profile region)
Fully developed flow region
Pressure
Entrance pressure drop
Linear pressure drop in fully developed flow region
0
L_e
x

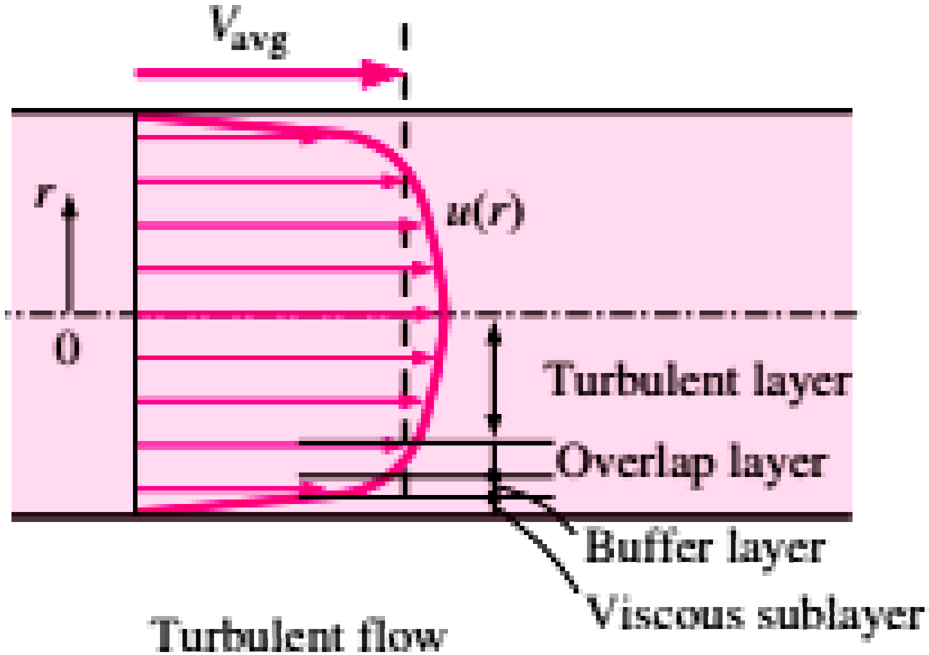
The fact that ***there is a nonzero pressure gradient along the horizontal pipe is a result of viscous effects***. ***If the viscosity were zero, the pressure would not vary with x***. **The need for the pressure drop can be viewed from two different standpoints**. **In terms of a force balance**, the pressure force is needed to overcome the viscous forces generated. **In terms of an energy balance**, the work done by the pressure force is needed to overcome the viscous dissipation of energy throughout the fluid. **If the pipe is not horizontal**, ***the pressure gradient along it is due in part to the component of weight in that direction***. This contribution due to the weight either enhances or retards the flow, depending on whether the flow is downhill or uphill.

The nature of the pipe flow is strongly dependent on whether the flow is laminar or turbulent. **The shear stress in laminar flow** ***is a direct result of momentum transfer among the randomly moving molecules (a microscopic phenomenon)***. **The shear stress in turbulent flow** ***is largely a result of momentum transfer among the randomly moving, finite-sized fluid particles (a macroscopic phenomenon)***. The net result is that the physical properties of the shear stress are quite different for laminar flow than for turbulent flow.

Turbulent Velocity Profile



Laminar flow



Turbulent flow

Typical velocity profiles for fully developed laminar and turbulent flows are given in Fig. Note that the velocity profile is parabolic in laminar flow but is much fuller in turbulent flow, with a sharp drop near the pipe wall. ***Turbulent flow along a wall can be considered to consist of four regions***, characterized by the distance from the wall. The very thin layer next to the wall where viscous effects are dominant is the **viscous** (or **laminar** or **linear** or **wall**) **sublayer**. The velocity profile in this layer is very nearly *linear*, and the flow is streamlined. Next to the viscous sublayer is the **buffer layer,** in which turbulent effects are becoming significant, but the flow is still dominated by viscous effects. Above the buffer layer is the

overlap (or **transition**) **layer**, also called the **inertial sublayer**, in which the turbulent effects are much more significant, but still not dominant. Above that is the **outer** (or **turbulent**) **layer** in the remaining part of the flow in which turbulent effects dominate over molecular diffusion (viscous) effects.

Entry Lengths

The hydrodynamic entry length is usually taken to be the distance from the pipe entrance to where the wall shear stress (and thus the friction factor) reaches within about 2 percent of the fully developed value. In ***laminar flow*, the hydrodynamic entry length** is given approximately as

$$\boxed{\frac{L_e}{d} \approx 0.06\,\mathrm{Re}_d \qquad \text{laminar}}$$

The maximum laminar entrance length, at $\mathrm{Re}_{d,\mathrm{crit}} = 2300$, is $L_e = 138d$, which is the longest development length possible.

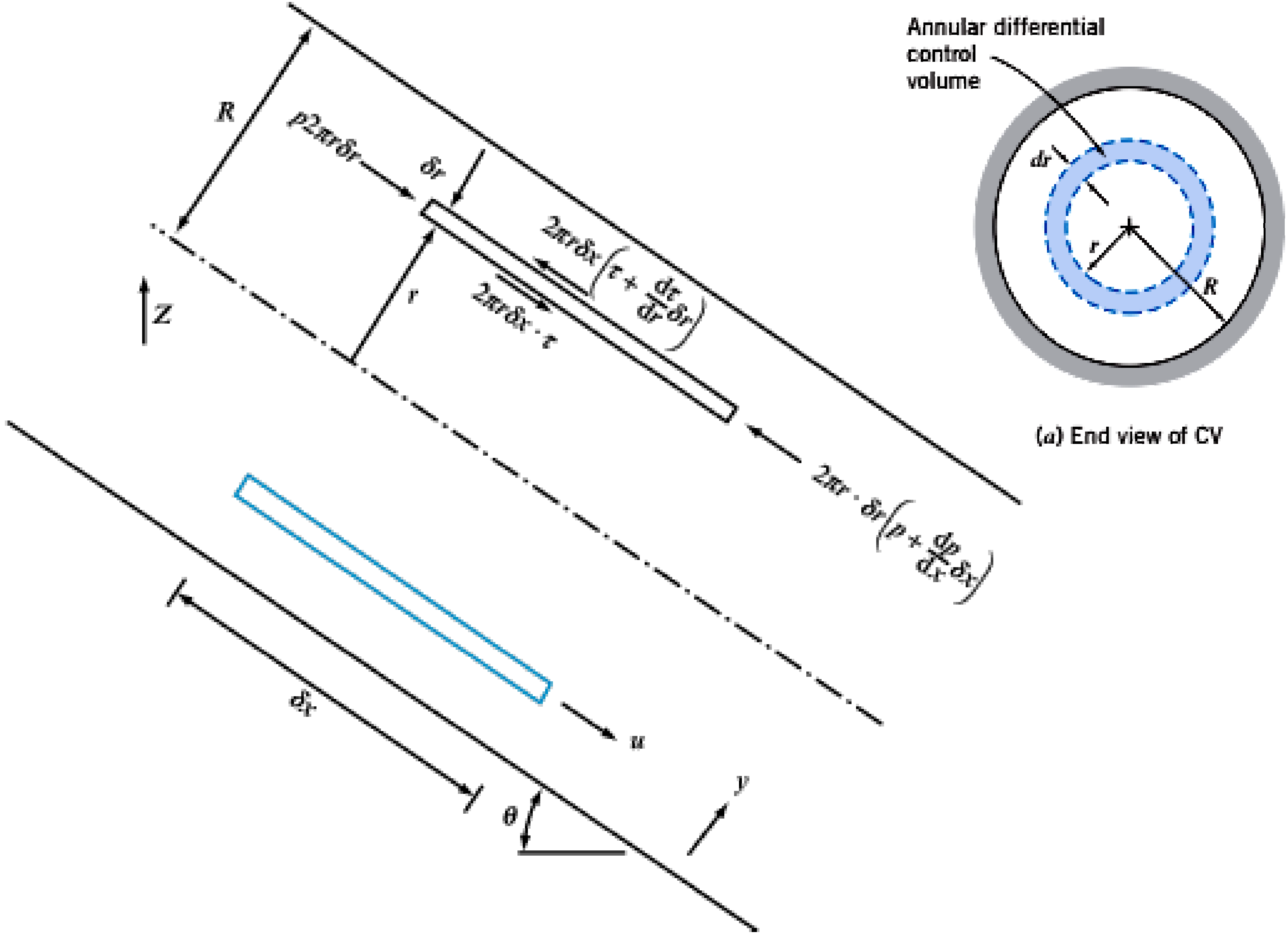
In turbulent flow, the boundary layers grow faster, and L_e is relatively shorter

$$\boxed{\frac{L_e}{d} \approx 1.6\,\mathrm{Re}_d^{1/4} \qquad \text{for} \quad \mathrm{Re}_d \leq 10^7}$$

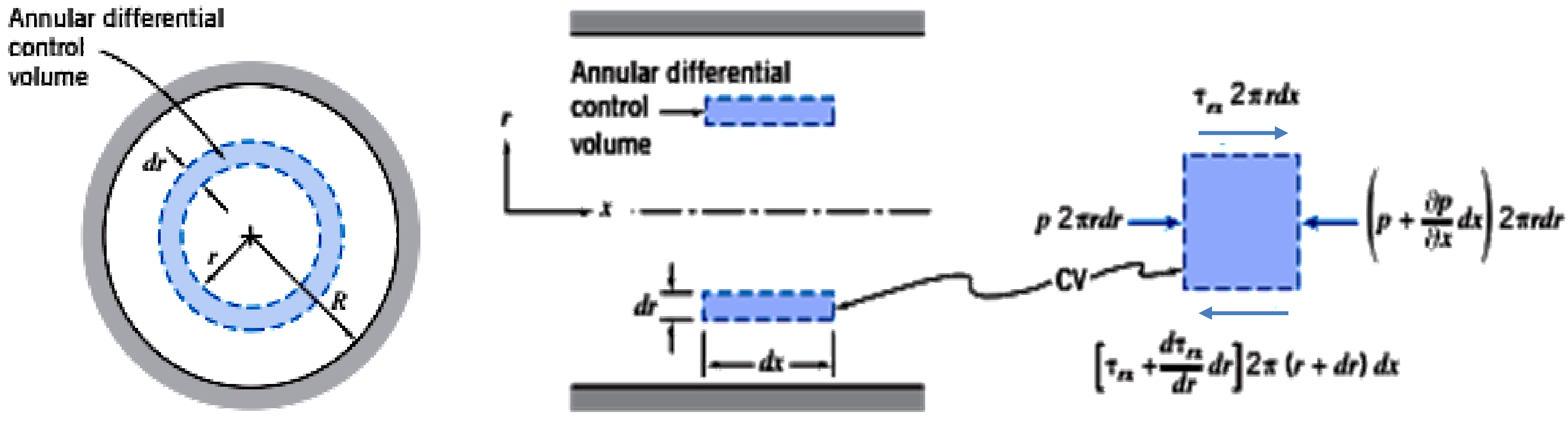
Laminar Flow In Pipes

In fully developed laminar flow, ***each fluid particle moves at a constant axial velocity along a streamline and the velocity profile u(r) remains unchanged in the flow direction***. ***There is no motion in the radial direction, and thus the velocity component in the direction normal to flow is everywhere zero***. ***There is no acceleration since the flow is steady and fully developed.***

Consider flow in a pipe, as shown in Fig. Now **consider a ring-shaped differential volume element of radius *r*, thickness *dr*, and length *dx* oriented coaxially with the pipe**. **In order to overcome the frictional force there must be pressure difference at the two ends of the element in the direction of the flow**. So, ***the volume element involves only pressure and viscous effects and thus the pressure and shear forces must balance each other***. Normal forces (pressure forces) act on the left and right ends of the control volume, and that tangential forces (shear forces) act on the inner and outer cylindrical surfaces.



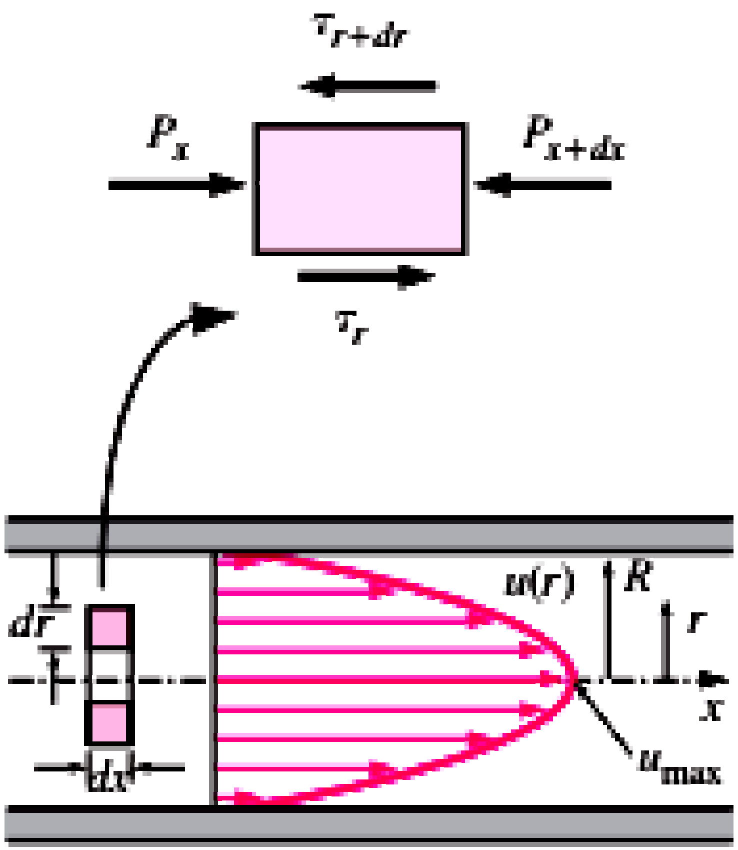
(a) End view of CV

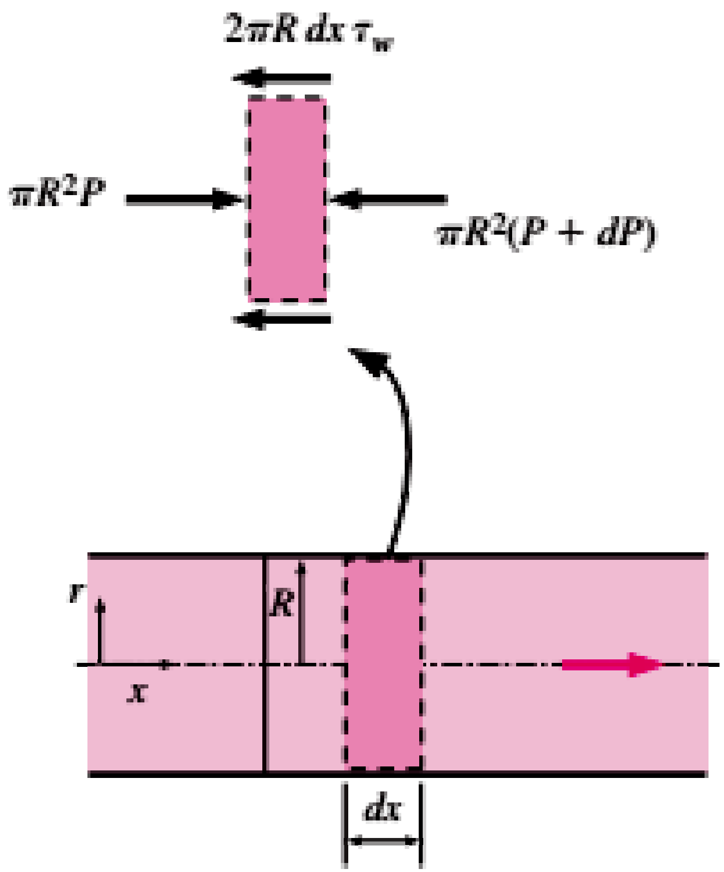


(a) End view of CV

(b) Side view of CV

(c) Forces on CV





A force balance on the volume element in the flow direction gives

$$(2\pi r\,dr\,P)_x - (2\pi r\,dr\,P)_{x+dx} + (2\pi r\,dx\,\tau)_r - (2\pi r\,dx\,\tau)_{r+dr} = 0$$

If the pressure at the left face of the control volume is p, then ***the pressure force on the left end is*** $dF_L = p2\pi r\,dr$

The pressure force on the right end is $dF_R = -\left(p + \frac{\partial p}{\partial x}dx\right)2\pi r\,dr$

If the shear stress at the inner surface of the annular control volume is τ_{rx}, then the ***shear force on the inner cylindrical surface*** is

$$dF_i = \tau_{rx}\,2\pi r\,dx$$

The shear force on the outer cylindrical surface is

$$dF_0 = -\left(\tau_{rx} + \frac{d\tau_{rx}}{\partial r}dr\right)2\pi(r+dr)dx$$

The sum of the *x* components of force, dF_L, dF_R, dF_I, *and* dF_O, acting on the control volume must be zero. This leads to the condition that (neglecting higher order terms)

$$-\frac{\partial p}{\partial x}2\pi r\,dr\,dx - \tau_{rx}2\pi\,dr\,dx - \frac{d\tau_{rx}}{dr}2\pi r\,dr\,dx = 0$$

The negative sign indicates that the pressure decreases in the direction of flow. Dividing this equation by *2πrdrdx* and solving for ∂p/∂x gives

$$-\frac{\partial p}{\partial x} = \frac{\tau_{rx}}{r} + \frac{d\tau_{rx}}{dr} = \frac{1}{r}\frac{d(r\tau_{rx})}{dr}$$

The left side of the equation is at most a function of *x* only (the pressure is uniform at each section); the right side is at most a function of *r* only (because the flow is fully developed). The equality must hold for any value of *r* and *x*, and an equality of the form *f(r) = g(x)* can be satisfied only if both *f(r)* and *g(x)* are equal to the same constant.

$$\frac{1}{r}\frac{d(r\tau_{rx})}{dr} = -\frac{\partial p}{\partial x} = constant \qquad \text{Or,} \qquad \frac{1}{r}\frac{d(r\tau_{rx})}{dr} = -\frac{\partial p}{\partial x}$$

In a constant diameter pipe, the pressure drops uniformly along the pipe length (except for the entrance region). Integrating this equation, we obtain

$$r\tau_{rx} = -\frac{r^2}{2}\left(\frac{\partial p}{\partial x}\right) + c_1 \qquad \text{Or,} \qquad \tau_{rx} = -\frac{r}{2}\left(\frac{\partial p}{\partial x}\right) + \frac{c_1}{r}$$

Let the radial distance of the fluid element from the boundary is equal to *y, so* $y = R - r$ *so* , $du/dy = -du/dr$

So, $\tau_{rx} = \mu du/dy = -\mu du/dr$, and we have

$$-\mu\frac{du}{dr} = -\frac{r}{2}\left(\frac{\partial p}{\partial x}\right) + \frac{c_1}{r} \quad \text{Or,} \quad \mu\frac{du}{dr} = \frac{r}{2}\left(\frac{\partial p}{\partial x}\right) - \frac{c_1}{r}$$

Again integrating, $$u = \frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) - \frac{c_1}{\mu}\ln r + c_2$$

Boundary conditions

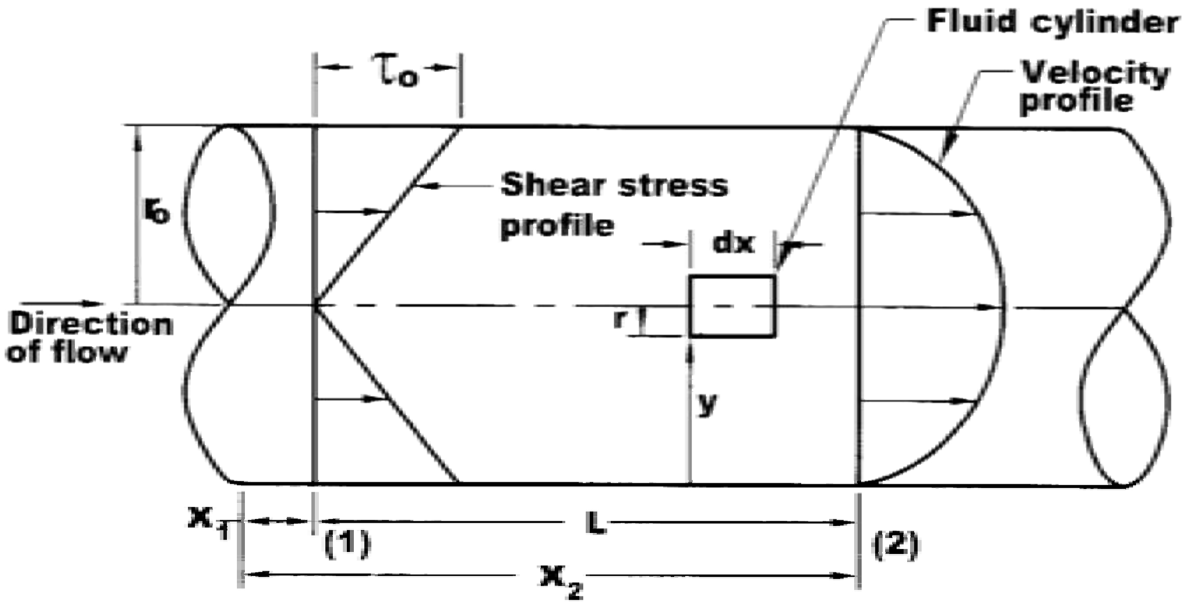
At $r = 0$, $\partial u/\partial r = 0$ *(because of symmetry about the centerline),* $c_1 = 0$

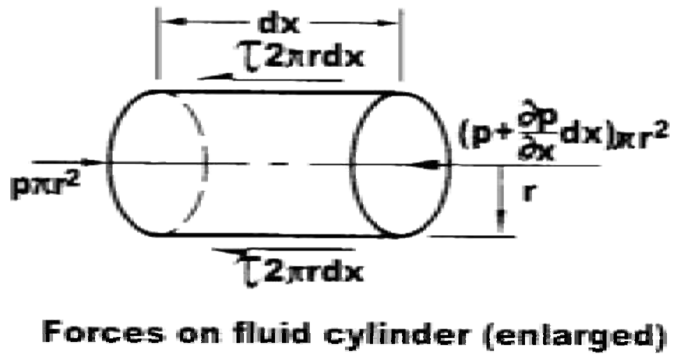
and at $r = R$, $u = 0$ *(the no-slip condition at the pipe surface).*

$$0 = \frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) + c_2 \qquad \text{Or,} \qquad c_2 = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)$$

and hence $$\boxed{u = \frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) - \frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)(r^2 - R^2)}$$

$$\boxed{u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]}$$ ***Velocity Profile***





$$\boxed{u(r) = -\frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)}$$

Therefore, **the velocity profile** in fully developed laminar flow in a pipe **is *parabolic* with a maximum at the centerline and minimum (zero) at the pipe wall**. Also, **the axial velocity *u* is positive for any *r***, and thus **the axial pressure gradient *dP*/*dx* must be negative** (i.e., **pressure must decrease in the flow direction because of viscous effects**).

Shear Stress Distribution

$$\boxed{u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{r}{R}\right)^2\right]}$$

and

$$\tau = \mu\frac{du}{dy} = -\mu\frac{du}{dr}$$

$$\boxed{\tau_{rx} = -\frac{r}{2}\left(\frac{\partial p}{\partial x}\right)}$$

At $r = 0$, the shear stress is zero i.e $\tau = 0$ and
at $r = R$, the shear stress is maximum i.e $\tau = \tau_{max} = - R/2\ (\partial p/\partial x) = \tau_{wall}$

Point of Maximum Velocity

$$\boxed{u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{r}{R}\right)^2\right]}$$

To find the point of maximum velocity, set *du/dr* equal to zero and solve for the corresponding *r*.

$$\frac{du}{dr} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)r$$

Thus, $\frac{du}{dr} = 0 \quad \text{at} \quad r = 0$ And **at** $r = 0$, $\boxed{u = u_{max} = U = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)}$

The maximum velocity occurs at the centerline. The velocity profile may be written in terms of the maximum (centerline) velocity as

$$\boxed{\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2}$$

The parabolic velocity profile, given by the Eq. for fully developed laminar pipe flow, was sketched in Fig.

Volume Flow Rate

The volume flow rate is $Q = \int_A \vec{V} \cdot d\vec{A} = \int_0^R u 2\pi r\, dr = \int_0^R \frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)(r^2 - R^2) 2\pi r\, dr$

$$\boxed{Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial x}\right)} \qquad \text{Or,} \qquad \boxed{Q = \frac{\pi}{2} u_{max} R^2}$$

Flow Rate as a Function of Pressure Drop

It is proved that in fully developed flow the pressure gradient, ∂p/∂x, is constant. Therefore, $\partial p/\partial x = (p_2 - p_1)/L = -\Delta p/L$. Substituting into Eq. for the volume flow rate gives

$$\boxed{Q = -\frac{\pi R^4}{8\mu}\left[\frac{-\Delta p}{L}\right] = \frac{\pi \Delta p R^4}{8\mu L} = \frac{\pi \Delta p D^4}{128\mu L}}$$

for laminar flow in a horizontal pipe.

Note that Q is a sensitive function of D; $Q \sim D^4$, so, for example, doubling the diameter D increases the flow rate Q by a factor of 16.

Average Velocity of Flow

The average velocity magnitude, V_{avg}, is given by

$$\boxed{\dot{m} = \rho V_{avg} A_c = \int_{A_c} \rho u(r)\, dA_c}$$

$$\boxed{V_{avg} = \frac{\int_{A_c} \rho u(r)\, dA_c}{\rho A_c} = \frac{\int_0^R \rho u(r) 2\pi r\, dr}{\rho \pi R^2} = \frac{2}{R^2} \int_0^R u(r) r\, dr}$$

$$\boxed{V_{avg} = \overline{V} = \frac{u_{max}}{2}}$$

Therefore, ***the average velocity in fully developed laminar pipe flow is one-half of the maximum velocity***.

The velocity profile is rewritten as

$$\boxed{u(r) = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right)}$$

Pressure Drop and Head Loss

$$-\partial p = \frac{8\mu V_{avg}}{R^2} \partial x$$

$$\overline{V} = \frac{Q}{A} = \frac{Q}{\pi R^2} = -\frac{R^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

Integrating this equation,

$$-\int_{p_1}^{p_2} \partial p = \frac{8\mu V_{avg}}{R^2} \int_{x_1}^{x_2} \partial x$$

$$(p_1 - p_2) = \frac{8\mu V_{avg}}{R^2}(x_2 - x_1) = \frac{8\mu V_{avg}}{R^2}L$$

Since $x_2 - x_1 = L$

$$\boxed{(p_1 - p_2) = \Delta p = \frac{32\mu V_{avg}}{D^2}L = \frac{128\mu L Q}{\pi D^4}}$$

where D is the diameter of pipe

The term $p_2 - p_1 = \Delta p =$ **Pressure drop**

This relation was discovered independently by Hagen (1*839)* and Poiseuille (1841), and is called the **Hagen-Poiseuille formula**. Using this equation, the viscosity of liquid can be obtained by measuring the pressure drop Δp.

A pressure drop due to viscous effects represents an **irreversible pressure** loss, and it is called ***pressure loss ΔP_L to emphasize that it is a loss (just like the head loss h_L***, which is proportional to it). the pressure drop is proportional to the viscosity μ of the fluid, and **ΔP would be zero if there were no friction. Therefore, the drop of pressure from P_1 to P_2 in this case is due entirely to viscous effects.**

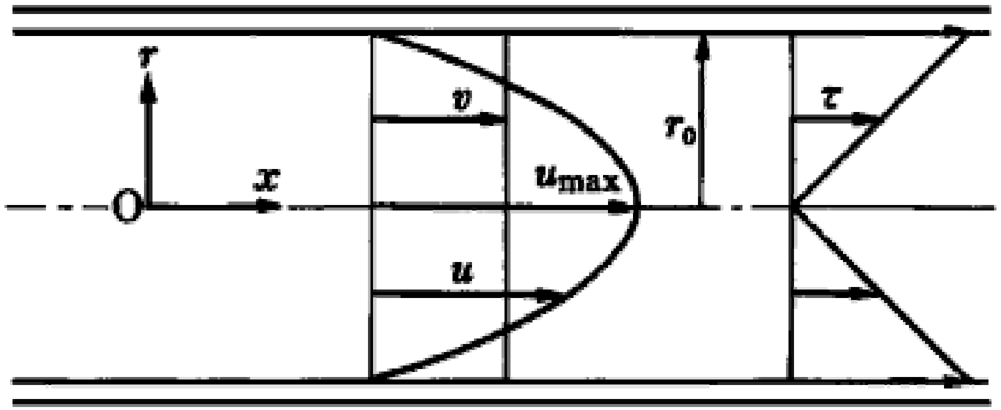
Three factors tend to reduce the pressure in a pipe flow: ***a decrease in pipe area***, ***an upward slope***, and ***friction***.

Let h_f = loss of pressure head
= $(p_1 - p_2)/\gamma$

Therefore,

$$\boxed{\frac{p_1 - p_2}{\gamma} = h_f = \frac{32\mu v L}{\gamma d^2}}$$

$$h_f = \frac{32LV^2}{gD\left(\frac{\rho V D}{\mu}\right)}$$



Inclined Pipes

Relations for inclined pipes can be obtained in a similar manner from a force balance in the direction of flow. The **only additional force** in this case ***is the component of the fluid weight in the flow direction***, whose magnitude is

$$W_x = W \sin\theta = \rho g V_{element} \sin\theta = \rho g(2\pi r\, dr\, dx) \sin\theta$$

Where θ is the angle between the horizontal and the flow direction.

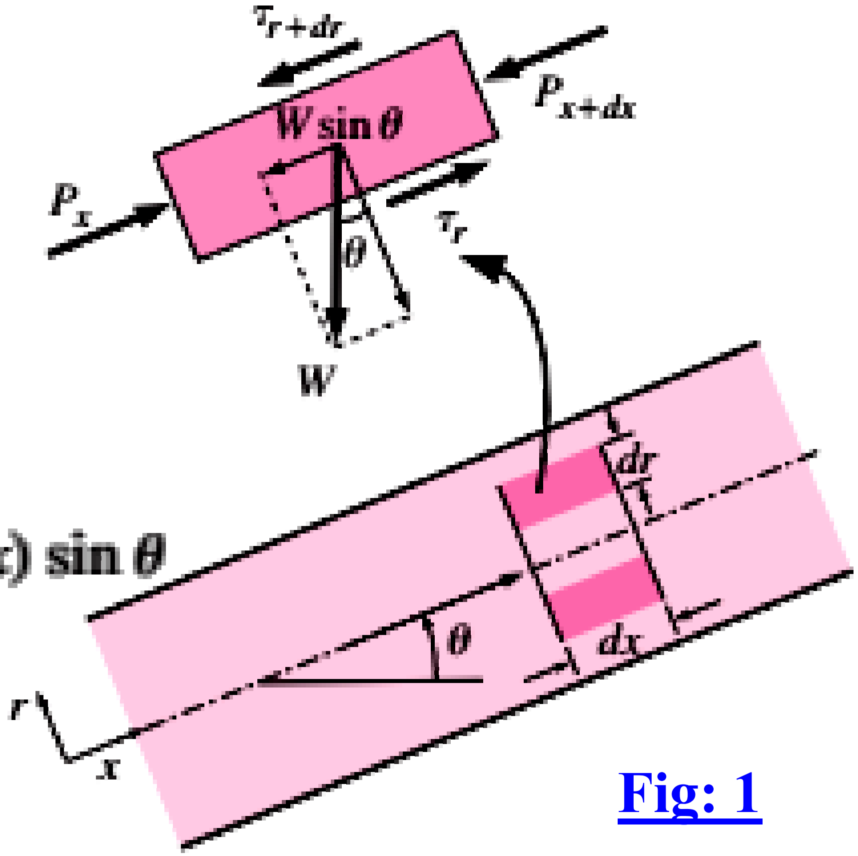
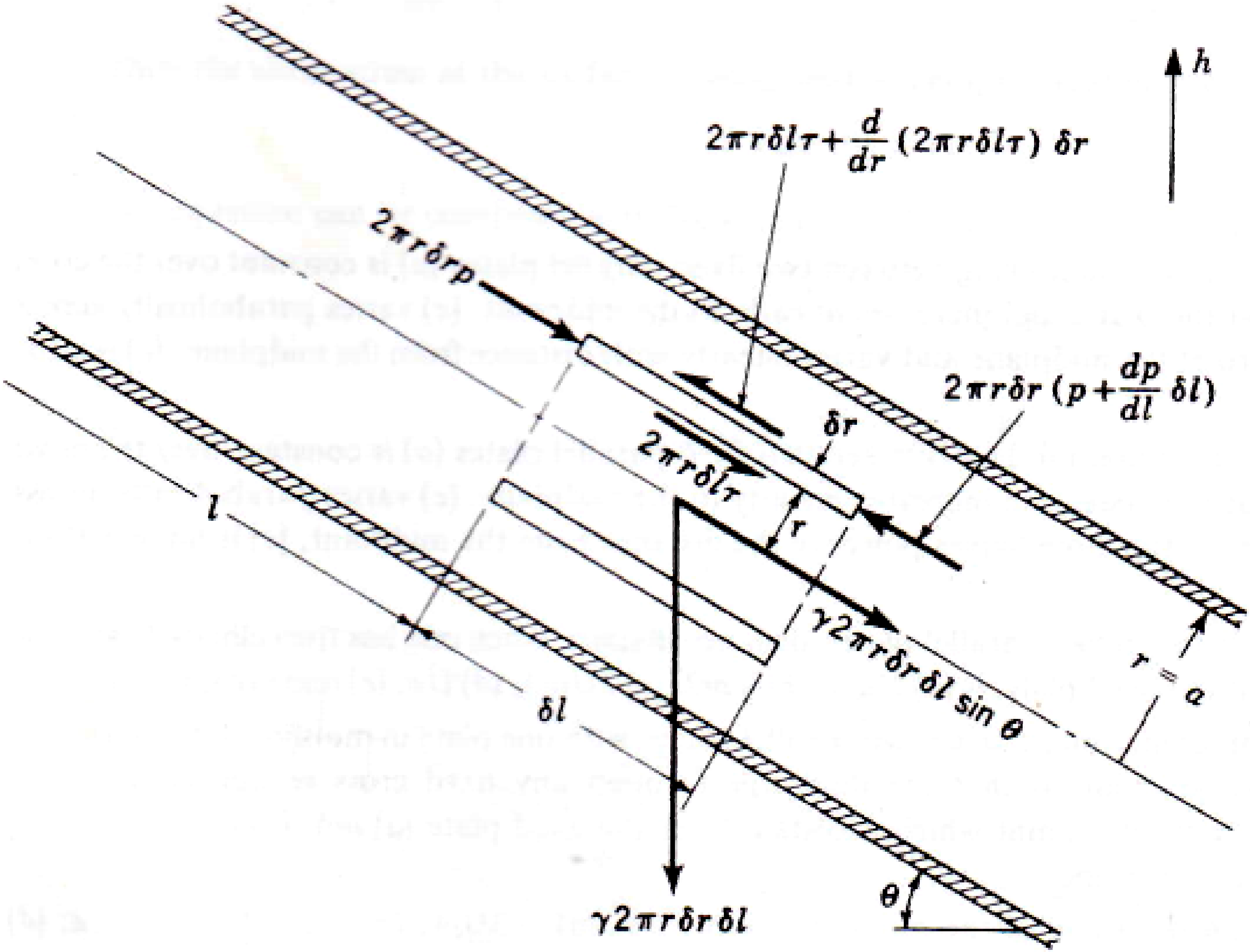


Fig: 1



h
$2\pi r\delta l\tau + \frac{d}{dr}(2\pi r\delta l\tau)\,\delta r$
$2\pi r\delta r p$
$2\pi r\delta r\,(p + \frac{dp}{dl}\,\delta l)$
δr
$2\pi r\delta l\tau$
r
l
$\gamma 2\pi r\delta r\,\delta l\,\sin\theta$
$r = a$
δl
θ
$\gamma 2\pi r\delta r\,\delta l$

Fig: 2

Consider an annular element in the flow of internal radius r and radial thickness δr, as shown in previous Fig. in an inclined tube, of radius R, carrying a fluid under laminar flow conditions. Applying the momentum equation to the situation we get

$$p 2\pi r \delta r - \left(p + \frac{dp}{dx}\delta x\right) 2\pi r \delta r + \tau 2\pi r\, dx$$
$$- \left[2\pi r \tau \delta x + \frac{d}{dr}(2\pi r \tau\, dx)\delta r\right] + W \sin\theta = 0,$$

where p is the flow static pressure, ***$W = mg$*** is the element weight and τ is the shear stress at radius r. Owing to the **assumption of steady, uniform conditions,** ***the flow acceleration is zero*** and, hence, ***the resultant force on the element is zero.***

Putting ***$W = 2\pi r \delta r \delta x \rho g$*** and **$\sin\theta = -dz/dx$**, where z is the elevation of the pipe above some horizontal datum, by dividing by $2\pi r \delta r \delta x$ reduces the above expression to

$$-\frac{dp}{dx} - \frac{1}{r}\frac{d}{dr}(r\tau) - \rho g \frac{dz}{dx} = 0 \qquad \text{Rearranging,} \qquad \frac{d}{dx}(p + \rho g z) + \frac{1}{r}\frac{d}{dr}(r\tau) = 0.$$

The term **(*p* + ρ*gz*)** is ***the flow piezometric pressure*** and ***is independent of r***, enabling the above equation to be integrated with respect to *r*. Hence,

$$\frac{r^2}{2}\frac{d}{dx}(p+\rho gh)+r\tau+C_1=0.$$

Let the radial distance of the fluid element from the boundary is equal to *y, so* $y = R - r$ *so* , $du/dy = -du/dr$

So, $\tau_{rx} = \mu du/dy = - \mu du/dr$, and we have

$$\tau=\mu\frac{du}{dy}=-\mu\frac{du}{dr}$$

and, by substituting for τ, above, $\frac{r^2}{2}\frac{d}{dx}(p+\rho gz)=r\mu\frac{du}{dr}-C_1$

and $$du=\left[\frac{r}{2\mu}\frac{d}{dx}(p+\rho gz)+\frac{C_1}{r\mu}\right]dr.$$

Integrating with respect to *r* yields $$u=\frac{r^2}{4\mu}\frac{d}{dx}(p+\rho gz)+\frac{C_1}{\mu}\log_e r+C_2.$$

Boundary conditions

At $r = 0$, $\partial u/\partial r = 0$ *(because of symmetry about the centerline),* $c_1 = 0$

and at $r = R$, $u = 0$ *(the no-slip condition at the pipe surface).*

$$C_2 = -\frac{R^2}{4\mu}\frac{d}{dx}(p+\rho gz)$$ and

$$\boxed{u = -\frac{(R^2-r^2)}{4\mu}\frac{d}{dx}(p+\rho gz).}$$ Or, $$\boxed{u(r) = -\frac{R^2}{4\mu}\left(\frac{dP}{dx}+\rho g\sin\theta\right)\left(1-\frac{r^2}{R^2}\right)}$$

The velocity profile may be seen to be parabolic. The negative sign is again present due to the fact that the pressure gradient will be negative in the flow direction.

The maximum velocity

To find the **point of maximum velocity, set *du/dr* equal to zero** and solve for the corresponding *r*.

$$\frac{du}{dr} = -\frac{r}{2\mu}\frac{d}{dx}(p+\rho gz)$$ Thus, $$\frac{du}{dr} = 0 \quad \text{at} \quad r = 0$$

And **at *r* = 0**, $$\boxed{u = u_{max} = -\frac{R^2}{4\mu}\frac{d}{dx}(p+\rho gz)}$$

The maximum velocity occurs at the centerline.

The volume flow rate

The incremental flow δQ through an annulus of radial width δr at radius r

$$\delta Q = u 2\pi r \delta r, \qquad \text{So,} \qquad Q = \int_0^R u 2\pi r \, \mathrm{d}r.$$

Substitution for u at general radius r yields an expression

$$Q = -\frac{\pi}{2\mu}\frac{\mathrm{d}}{\mathrm{d}x}(p+\rho g z)\int_0^R (R^2 r - r^3)\,\mathrm{d}r$$

$$= -\frac{\pi}{2\mu}\frac{\mathrm{d}}{\mathrm{d}x}(p+\rho g z)\left(R^2\frac{r^2}{2} - \frac{r^4}{4}\right)_0^R$$

$$= -\frac{\pi}{8\mu}\frac{\mathrm{d}}{\mathrm{d}x}(p+\rho g z)R^4$$

For Fig: 2

In terms of a pressure drop Δp over a length l of pipe of diameter d,

$$\boxed{Q = \Delta p \pi d^4 / 128 \mu L}$$

For Fig: 2

$$\boxed{\dot{V} = \frac{(\Delta P - \rho g L \sin\theta)\pi D^4}{128 \mu L}}$$

For Fig: 1

The mean flow velocity

$V_{avg} = Q/A$, where A is the pipe cross-sectional area $\pi d^2/4$. Hence,

$$\boxed{\bar{u} = -\frac{\pi}{8\mu}\frac{d}{dx}(p+\rho g z)R^2 = \frac{1}{2}u_{max}}$$

For Fig: 2

$$\boxed{V_{avg} = \frac{(\Delta P - \rho g L \sin\theta)D^2}{32\mu L}}$$

For Fig: 1

The pressure loss

$$\boxed{\Delta p = 128\mu l Q/\pi d^4.}$$

Hagen–Poiseuille equation

Alternatively, substituting for $Q = (\pi d^2/4)V_{avg}$ so, $\boxed{\Delta p = 32\mu l\bar{u}/d^2.}$

The shear stress distribution

$$\boxed{u = -\frac{(R^2-r^2)}{4\mu}\frac{d}{dx}(p+\rho g z)}$$ and, $\tau = \mu\frac{du}{dy} = -\mu\frac{du}{dr}$ So, $$\boxed{\tau = -\frac{r}{2}\frac{d(p+\rho g z)}{dx}}$$

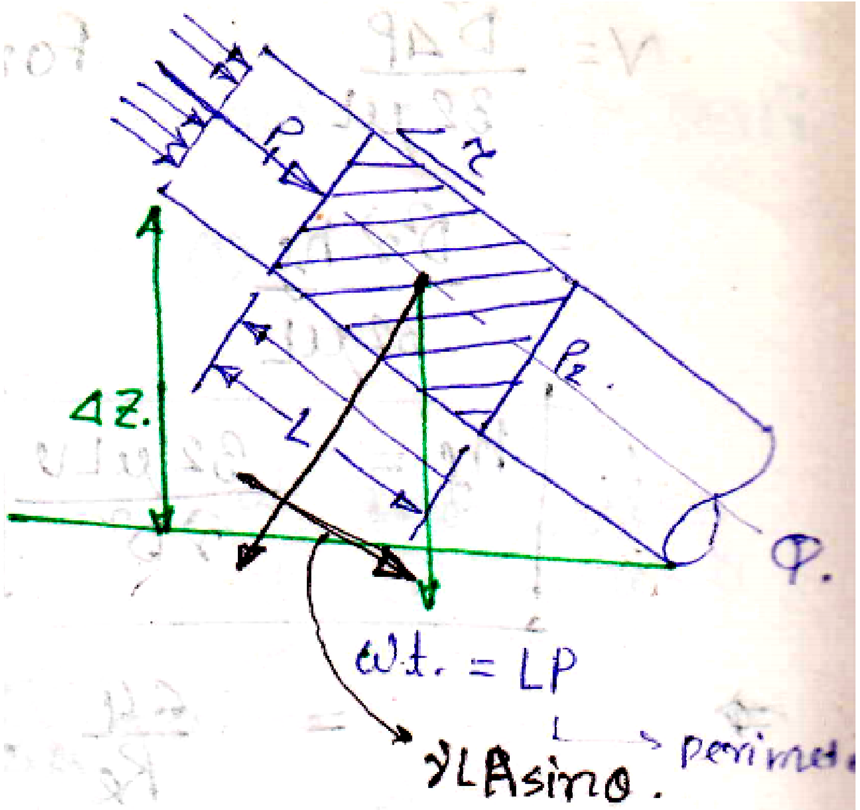
At $r = 0$, (pipe centre) the shear stress is zero i.e $\tau = 0$ and
at $r = R$, (pipe wall) the shear stress is maximum i.e

$$\boxed{\tau = \tau_{wall} = \tau_{max} = -\frac{R}{2}\frac{d(p+\rho g z)}{dx}}$$

Now, $P_1 = p_1 + \rho g z_1$, $P_2 = p_2 + \rho g z_2$ and $\partial P/\partial x = (P_2 - P_1)/L = -\Delta p/L$

The pressure drop Δp over a length L of a horizontal section of pipe is

$$\boxed{\Delta P = \frac{2\tau_{wall} L}{R}}$$



$$P_1 = p_1 + \rho g z_1 \quad \text{Or,} \quad \frac{P_1}{\rho g} = \frac{p_1}{\rho g} + z_1 = HGL_1$$

$$P_2 = p_2 + \rho g z_2 \quad \text{Or,} \quad \frac{P_2}{\rho g} = \frac{p_2}{\rho g} + z_2 = HGL_2$$

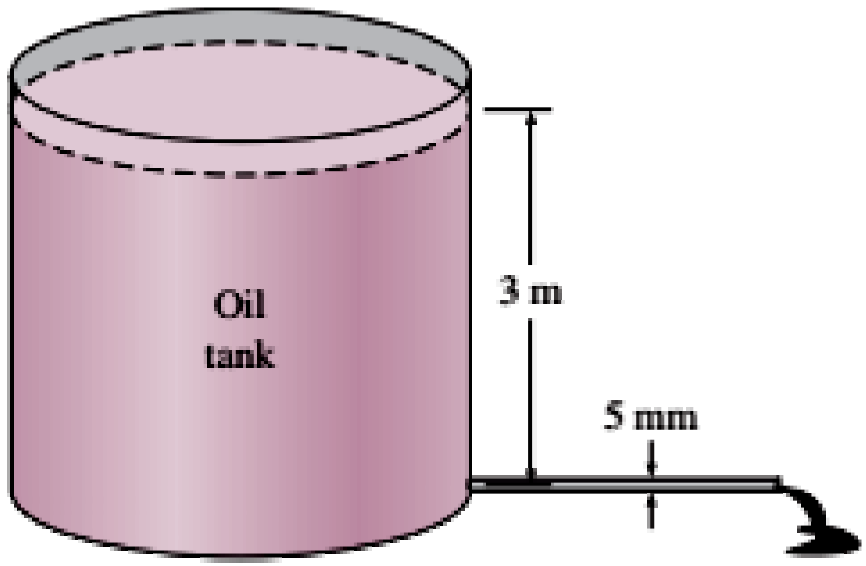
If **$HGL_1 > HGL_2$, then the flow will be in the direction from point 1 to point 2**. $HGL_1 < HGL_2$, then the flow will be in the direction from point 2 to point 1.

All the relations for inclined pipe are identical to the corresponding relations for horizontal pipes, except that ΔP is replaced by $\Delta P \pm \rho g L \sin\theta$. Note that $\theta > 0$ and thus $\sin\theta > 0$ (and for fig.1 it is downhill flow, while for fig. 2 it is uphill flow) and $\theta < 0$ and thus $\sin\theta < 0$, vise versa.

In inclined pipes, ***the combined effect of pressure difference and gravity drives the flow***. ***Gravity helps downhill flow but opposes uphill flow***. Therefore, **much greater pressure differences need to be applied to maintain a specified flow rate in uphill flow** although this becomes important only for liquids, because the density of gases is generally low. In the special case of *no flow* ($Q = 0$), we have $\Delta P = \rho g L \sin\theta$, which is what we would obtain from fluid statics.

Problem:

Oil with a density of 850 kg/m^3 and kinematic viscosity of 0.00062 m^2/s is being discharged by a 5-mm-diameter, 40-m-long horizontal pipe from a storage tank open to the atmosphere. The height of the liquid level above the center of the pipe is 3 m. Disregarding the minor losses, determine the flow rate of oil through the pipe.



Assumptions

1. The flow is steady and incompressible.
2. The entrance effects are negligible, and thus the flow is fully developed.
3. The entrance and exit loses are negligible.
4. The flow is laminar (to be verified).
5. The pipe involves no components such as bends, valves, and connectors.
6. The piping section involves no work devices such as pumps and turbines.

Properties

For oil, $\rho = 850 \text{ kg/m}^3$ and $\nu = 0.00062 \text{ m}^2\text{/s}$,

$$\mu = \rho\nu = (850 \text{ kg/m}^3)(0.00062 \text{ m}^2 / \text{s}) = 0.527 \text{ kg/m·s}$$

Analysis: The pressure at the bottom of the tank is

$$P_{1,gage} = \rho g h$$

$$= (850 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(3 \text{ m})\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right)$$

$$= 25.02 \text{ kN/m}^2$$

Disregarding inlet and outlet losses, ***the pressure drop across the pipe*** is

$$\Delta P = P_1 - P_2 = P_1 - P_{atm} = P_{1,gage} = 25.02 \text{ kN/m}^2 = 25.02 \text{ kPa}$$

The flow rate through a horizontal pipe in laminar flow is determined from

$$\boxed{\dot{V}_{horiz} = \frac{\Delta P \pi D^4}{128 \mu L} = \frac{(25.02 \text{ kN/m}^2)\pi(0.005 \text{ m})^4}{128(0.527 \text{ kg/m}\cdot\text{s})(40 \text{ m})}\left(\frac{1000 \text{ kg}\cdot\text{m/s}^2}{1 \text{ kN}}\right) = \mathbf{1.821\times10^{-8}\ m^3/s}}$$

The average fluid velocity and the Reynolds number in this case are

$$V = \frac{\dot{V}}{A_c} = \frac{\dot{V}}{\pi D^2/4} = \frac{1.821\times10^{-8} \text{ m}^3/\text{s}}{\pi(0.005 \text{ m})^2/4} = 9.27\times10^{-4} \text{ m/s}$$

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{(850 \text{ kg/m}^3)(9.27\times10^{-4} \text{ m/s})(0.005 \text{ m})}{0.527 \text{ kg/m}\cdot\text{s}} = 0.0075$$

which is less than 2300. Therefore, the flow is laminar and the analysis above is valid.

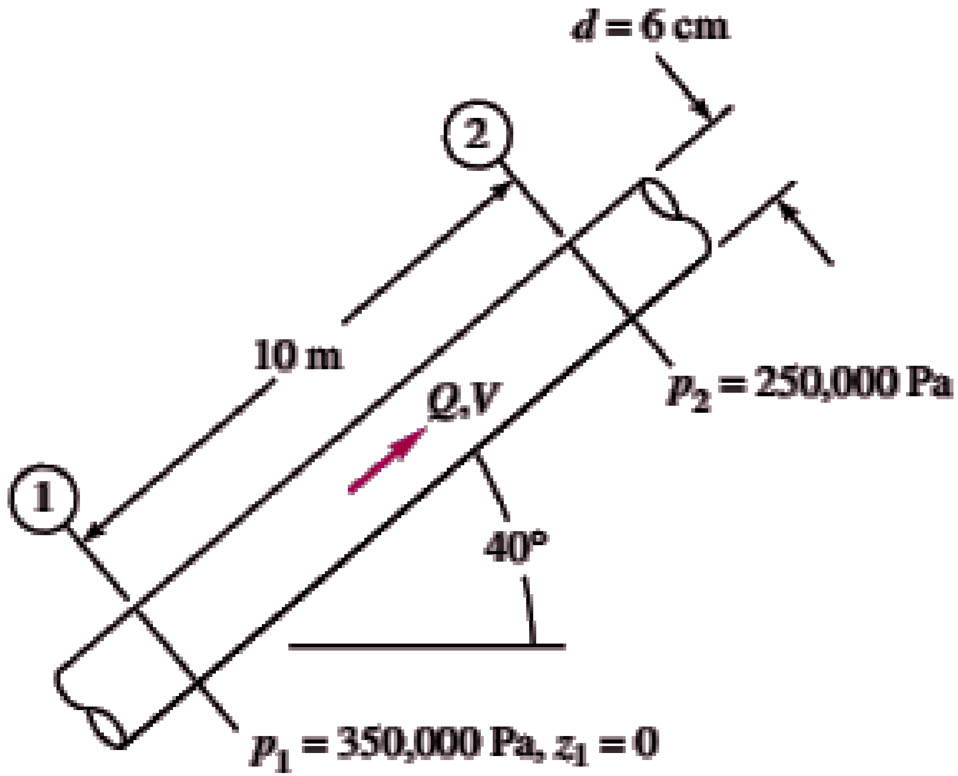
Problem:

An oil with $\rho = 900\ kg/m^3$ and $\nu = 0.0002\ m^2/s$ flows upward through an inclined pipe as shown in Fig. The pressure and elevation are known at sections 1 and 2, 10m apart. Assuming steady laminar flow, **(*a*)** verify that the flow is up, **(*b*)** compute h_f between 1 and 2, and compute **(*c*)** Q, **(*d*)** V, and **(*e*)** Re_d. Is the flow really laminar?

Solution:

$$\mu = \rho\nu = (900\ kg/m^3)(0.0002\ m^2/s)$$
$$= 0.18\ kg/m \cdot s$$

$$z_2 = \Delta L \sin 40^0 = (10\ m)(0.643)$$
$$= 6.43\ m$$



(a) The flow goes in the direction of falling HGL; therefore compute the hydraulic grade-line height at each section:

$$HGL_1 = z_1 + \frac{p_1}{\rho g} = 0 + \frac{350{,}000}{900(9.807)} = 39.65 \text{ m}$$

$$HGL_2 = z_2 + \frac{p_2}{\rho g} = 6.43 + \frac{250{,}000}{900(9.807)} = 34.75 \text{ m}$$

The HGL is lower at section 2; hence the flow is up from 1 to 2 as assumed.

(b) The head loss is the change in HGL:

$\mathbf{h_f = HGL_1 - HGL_2}$ = 39.65m - 34.75m **= 4.9 m**

Half the length of the pipe is quite a large head loss.

(c) The flow rate Q from the various laminar flow formulas

$$\boxed{Q = \frac{\pi \rho g d^4 h_f}{128 \mu L} = \frac{\pi(900)(9.807)(0.06)^4(4.9)}{128(0.18)(10)} = 0.0076 \text{ m}^3/\text{s}}$$

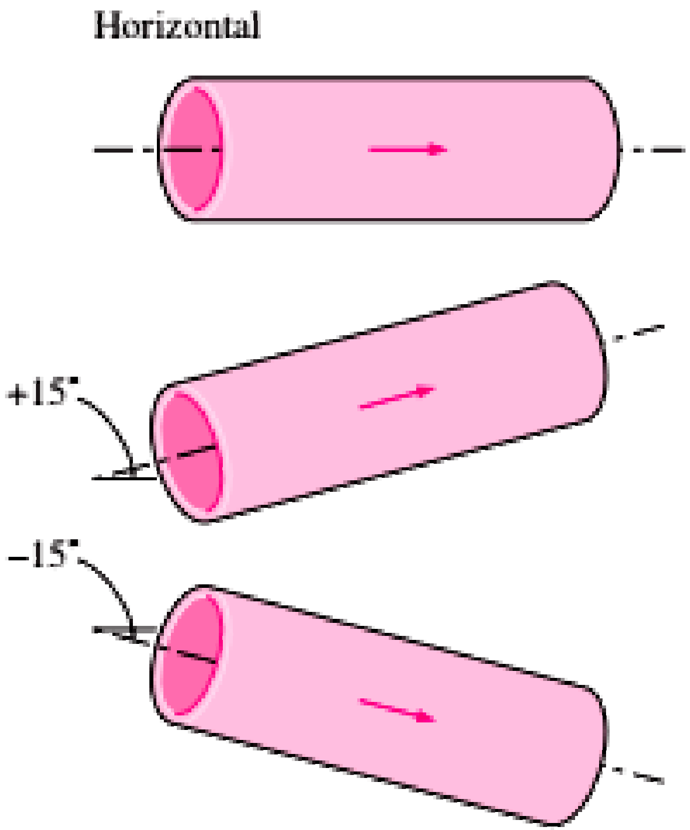
(d) the average velocity $\boxed{V = \frac{Q}{\pi R^2} = \frac{0.0076}{\pi(0.03)^2} = 2.7\text{ m/s}}$

(e) the Reynolds number is $\boxed{\mathrm{Re}_d = \frac{Vd}{\nu} = \frac{2.7(0.06)}{0.0002} = 810}$

This is well below the transition value **$\mathrm{Re}_d < 2300$**, so we are fairly certain **the flow is laminar**.

Problem:

Oil at 20°C (ρ = 888 kg/m^3 and μ = 0.800 kg/m·s) is flowing steadily through a 5cm-diameter 40m-long pipe. The pressure at the pipe inlet and outlet are measured to be 745 and 97 kPa, respectively. Determine the flow rate of oil through the pipe assuming the pipe is ***(a)*** horizontal, ***(b)*** inclined 15° upward, ***(c)*** inclined 15° downward. Also verify that the flow through the pipe is laminar.



Properties

Given: $\rho = 888\ kg/m^3$ and $\mu = 0.800\ kg/m \cdot s$,
Pipe dia, $D = 5cm = 0.05m$ and pipe length, $L = 40m$
$p_1 = 745\ kPa$ and $p_2 = 97\ kPa$,

Assumptions

1. The flow is steady and incompressible.
2. The entrance effects are negligible, and thus the flow is fully developed.
3. The pipe involves no components such as bends, valves, and connectors.
4. The piping section involves no work devices such as a pump or a turbine.

Analysis

The pressure drop across the pipe and the pipe cross-sectional area are

$$\Delta P = p_1 - p_2 = 745 - 97 = 648\ kPa \text{ and}$$

$$A_c = \pi D^2/4 = \pi(0.05\ m)^2/4 = 0.001963\ m^2$$

The flow rate for all three cases can be determined from Eq,

$$\boxed{\dot{V} = \frac{(\Delta P - \rho g L \sin\theta)\pi D^4}{128\mu L}}$$

According to Fig: 1

where θ is the angle the pipe makes with the horizontal.

(a) For ***the horizontal pipe flow*** case, ***θ = 0*** and thus ***sinθ = 0***. Therefore,

$$\dot{V}_{\text{horiz}} = \frac{\Delta P\,\pi D^4}{128\mu L} = \frac{(648\ \text{kPa})\pi(0.05\ \text{m})^4}{128(0.800\ \text{kg/m}\cdot\text{s})(40\ \text{m})}\left(\frac{1000\ \text{N/m}^2}{1\ \text{kPa}}\right)\left(\frac{1\ \text{kg}\cdot\text{m/s}^2}{1\ \text{N}}\right)$$

$$= \mathbf{0.00311\ m^3/s}$$

(*b*) ***inclined 15° upward***, we have ***θ = +15°***

$$\dot{V}_{\text{uphill}} = \frac{(\Delta P - \rho g L \sin\theta)\pi D^4}{128\mu L}$$

$$= \frac{[648{,}000\ \text{Pa} - (888\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(40\ \text{m})\sin 15°]\pi(0.05\ \text{m})^4}{128(0.800\ \text{kg/m}\cdot\text{s})(40\ \text{m})}\left(\frac{1\ \text{kg}\cdot\text{m/s}^2}{1\ \text{Pa}\cdot\text{m}^2}\right)$$

$$= \mathbf{0.00267\ m^3/s}$$

(*c*) For ***downhill flow with an inclination of 15°***, we have ***θ = -15°*** and

$$\dot{V}_{\text{downhill}} = \frac{(\Delta P - \rho g L \sin\theta)\pi D^4}{128\mu L}$$

$$= \frac{[648{,}000\ \text{Pa} - (888\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(40\ \text{m})\sin(-15°)]\pi(0.05\ \text{m})^4}{128(0.800\ \text{kg/m}\cdot\text{s})(40\ \text{m})}\left(\frac{1\ \text{kg}\cdot\text{m/s}^2}{1\ \text{Pa}\cdot\text{m}^2}\right)$$

$$= \mathbf{0.00354\ m^3/s}$$

The flow rate is the highest for the downhill flow case, as expected. The average fluid velocity and the Reynolds number in this case are

$$V_{avg} = \frac{\dot{V}}{A_c} = \frac{0.00354\ \text{m}^3/\text{s}}{0.001963\ \text{m}^2} = 1.80\ \text{m/s}$$

$$\text{Re} = \frac{\rho V_{avg} D}{\mu} = \frac{(888\ \text{kg/m}^3)(1.80\ \text{m/s})(0.05\ \text{m})}{0.800\ \text{kg/m}\cdot\text{s}} = 100$$

which is much less than 2300. Therefore, **the flow is *laminar*** for all ***three cases and the analysis is valid***.

Problem:

Lubricating oil of specific gravity 0.82 and dynamic viscosity 12.066 x10^{-2} N.s/m^2 [1.23 x 10^{-2} kg*(f)*-s/m^2] is pumped at a rate of 0.02 m^3/s through a 0.15 m diameter 300m long pipe. ***Calculate pressure drop, average shear stress at the wall of the pipe and the power required to maintain flow*** **(a)** if the pipe is horizontal; **(b)** if the pipe is inclined at 15 degrees with the horizontal and the flow is **(i)** in the upward direction, **(ii)** in the downward direction. Also ***determine the slope of the pipe and the direction of flow so that pressure gradient along the pipe is zero***.

Properties

Given: $SG = 0.82$ and $\mu = 12.006 \times 10^{-2}$ N.s/m^2, Flow rate, Q = 0.02 m^3/s,
Pipe dia, D = 0.15m and pipe length, L = 300m.

Assumptions

1. The flow is steady and incompressible.
2. The entrance effects are negligible, and thus the flow is fully developed.

Analysis

The average flow velocity, $V_{avg} = Q/A$ so,

$$V = \frac{0.02}{\frac{\pi}{4}(0.15)^2} = 1.132 \text{ m/s}$$

(a) For **horizontal pipe** from ***Hagen-Poiseuille equation***, we have

$$\boxed{(p_1 - p_2) = \Delta p = \frac{32\mu V_{avg}}{D^2} L = \frac{128\mu L Q}{\pi D^4}}$$

$$(P_1 - P_2) = \frac{32 \times 12.066 \times 10^{-2} \times 1.132 \times 300}{(0.15)^2} = 58\,277.17 \text{ N/m}^2$$

And $\tau_{max} = -R/2\,(\partial p/\partial x) = \tau_{wall}$

$$\tau_0 = \left(-\frac{\partial p}{\partial x}\right)\frac{R}{2} = \frac{58277.17}{300} \times \frac{0.15}{2 \times 2} = 7.285 \text{ N/m}^2$$

Power required to maintain the flow is

$$P = Q(p_1 - p_2)$$
$$= 0.02 \times 58\,277.17 = 1166 \text{ W} = 1.166 \text{ kW}$$

(b) (i) For **inclined pipe with flow in upward direction**, we have

$$\boxed{\Delta p = 32\mu l\bar{u}/d^2.}$$

$$\Delta p = P_1 - P_2 = (p_1 + \rho g z_1) - (p_2 + \rho g z_2) = \gamma(h_1 - h_2)$$

so $$(h_1 - h_2) = \frac{32 \times 12.066 \times 10^{-2} \times 1.132 \times 300}{(9810 \times 0.082) \times (0.15)^2} = 7.245 \text{ m}$$

Or, $$\left(\frac{p_1}{w} + 0\right) - \left(\frac{p_2}{w} + Z_2\right) = 7.245$$

Since $$\frac{Z_2}{300} = \sin 15° = 0.2588 \,;\, Z_2 = 77.64 \text{ m}$$

$$(p_1 - p_2) = (9\,810 \times 0.82)(7.245 + 77.64)$$
$$= 682\,832 \text{ N/m}^2 = 682.832 \text{ kN/m}^2$$

$$\boxed{\tau = \tau_{wall} = \tau_{max} = -\frac{R}{2}\frac{d(p+\rho gz)}{dx}}$$

$$\tau_0 = w\left(-\frac{\partial h}{\partial x}\right)\frac{R}{2}$$

$$= \frac{(9810\times 0.82)\times 7.245}{300}\times\frac{0.15}{2\times 2} = 7.285\ \text{N/m}^2$$

Power required to maintain the flow is

$$P = Q(p_1 - p_2)$$
$$= 0.02\times 682\,832 = 13\,657\ \text{W} = 13.657\ \text{kW}$$

(ii) For **<u>inclined pipe with flow in downward direction</u>**, we have

$$w(h_1 - h_2) = \frac{32\mu VL}{D^2}$$

So,

$$(h_1 - h_2) = \frac{32\times 12.066.10^{-2}\times 1.132\times 300}{(9810\times 0.82)\times(0.15)^2} = 7.245\ \text{m}$$

$$\left(\frac{p_1}{w}+Z_1\right)\left(\frac{p_2}{w}+0\right) = 7.245$$

Since $$\frac{Z_1}{300} = \sin 15^\circ = 0.2588;\ Z_1 = 77.64 \text{ m}$$

$$(p_1 - p_2) = (9810 \times 0.82)(7.245 - 77.64)$$
$$= -566\,271 \text{N/m}^2 = -566.271 \text{ kN/m}^2$$

In this case the pressure increases in the direction of flow, or ***there is a positive pressure gradient.***

$$\tau_0 = w\left(-\frac{\partial h}{\partial x}\right)\frac{R}{2}$$

$$= \frac{(9810 \times 0.82) \times 7.245}{300} \times \frac{0.15}{2 \times 2} = 7.285 \text{ N/m}^2$$

In this case the resistance to flow is compensated by the excessive downward slope of the pipe and hence no external power is required to maintain the flow. Moreover, in this case the flow will have to be regulated by means of a regulating value to maintain the given flow rate.

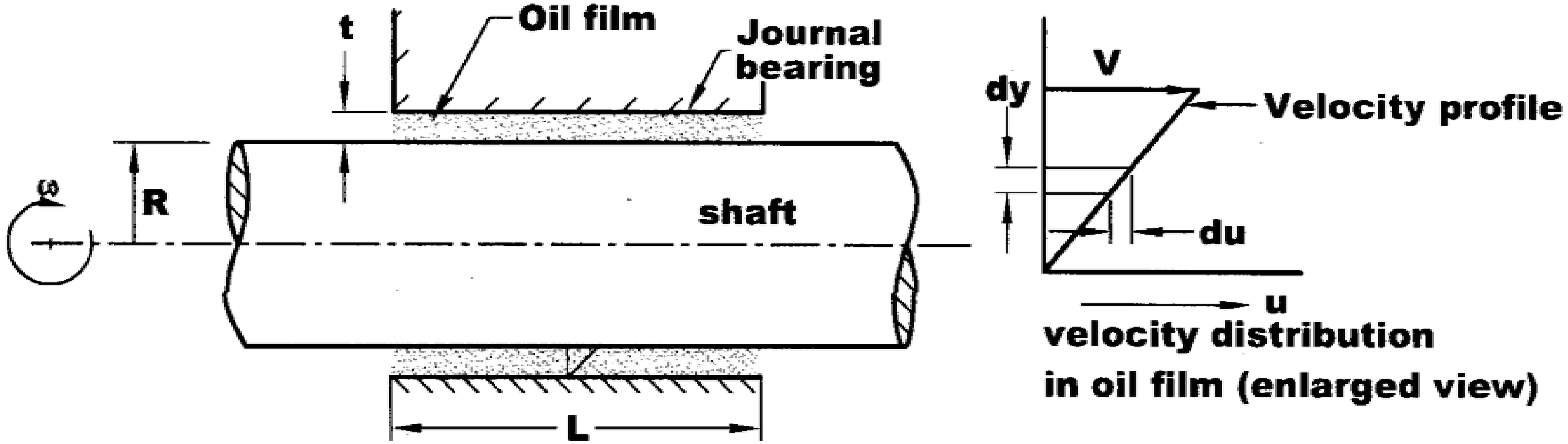
(C) For **pressure gradient along the pipe to be zero**, $P_1 = P_2$

Then from the two above noted cases of inclination of the pipe we have either, $-z_2 = 7.245$m, or $z_1 = 7.245$m, from which it may be concluded that point 1 is higher than point 2, so that the flow is in the downward direction. The slope required to be provided for the pipe in this case is given by

$$\boxed{\begin{aligned} \sin\theta &= \frac{Z_1}{300} = \frac{7.245}{300} = 0.0242 \\ \theta &= 1°23' \end{aligned}}$$

Problem:

A shaft is rotating in a journal bearing of length L. The clearance between the shaft and the bearing is filled with oil of viscosity μ. The thickness of film is t. If the diameter of the shaft is D, **find an expression for the power absorbed in overcoming viscous resistance.**



Assumptions :

- Oil film thickness is very small
- Velocity profile in oil film is linear

Nomenclature :

L – length of bearing, t – thickness of oil film, N – rpm of shaft

D – diameter of shaft, R – radius of shaft, μ - viscosity of oil

Derivation of Equation :

Angular speed of shaft, $\omega = \frac{2\pi N}{60}$

Peripheral speed of the shaft,

$$V = R\omega$$

or, $$V = R \times \frac{2\pi N}{60}$$

or, $$V = \frac{D}{2} \times \frac{2\pi N}{60}$$

$$V = \frac{\pi DN}{60} \quad \text{------ (1)}$$

From Newton's law of viscosity shear stress is given by

$$\tau = \mu \frac{du}{dy}$$

Now, $$\frac{du}{dy} = \frac{V - 0}{t}$$

or, $$\frac{du}{dy} = \frac{V}{t}$$

or, $$\frac{du}{dt} = \frac{\pi DN}{60t}$$

Therefore, $\tau = \mu \dfrac{\pi DN}{60t}$

Shear stress is given by

$$F = \tau \times \text{ area of surface of shaft}$$

$$\text{or, } F = \frac{\mu\pi DN}{60t} \times \pi DL$$

$$\text{or, } F = \frac{\mu\pi^2 D^2 NL}{60t}$$

Torque required to turn the shaft

$$T = \text{shear force} \times \text{radius of shaft}$$

$$\text{or, } T = \frac{\mu\pi^2 D^2 NL}{60t} \times \frac{D}{2}$$

$$\text{or, } T = \frac{\mu\pi^2 D^3 NL}{120t}$$

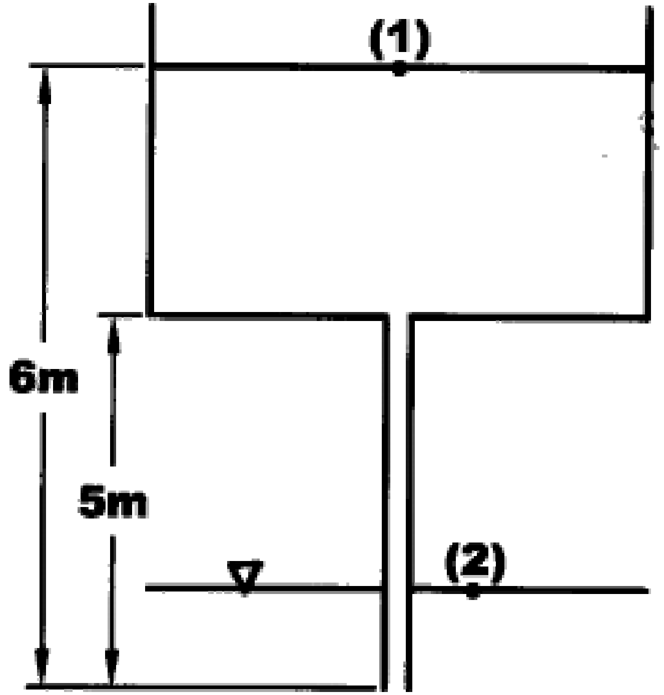
Therefore, power absorbed in overcoming the friction, **$P = T \times \omega$**

$$\text{or, } P = \frac{\mu\pi^2 D^3 NL}{120t} \times \frac{2\pi N}{60}$$

$$\text{or, } P = \frac{\mu\pi^3 D^3 N^2 L}{3600t}$$

Problem:

A certain amount of liquid is flowing from a reservoir through a tube of 6 mm diameter. The specific weight of the liquid is 8.64 kN/m^3 and its viscosity is 0.01 Ns/m^2. Neglecting minor losses find the flow rate.



Given Data :

Diameter of tube, d = 6 mm

Specific weight of water, $\gamma = 8.64\, kN/m^3$

Viscosity of liquid, $\mu = 0.01\, Ns/m^2$

Solution :

Applying Bernoulli's equation between points (i) and (ii),

$$\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + Z_1 = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + Z_2 + h_f$$

or, $0 + 0 + 6 = 0 + 0 + 0 + h_f$

or, $h_f = 6m$

Let v = velocity of liquid through the tube

Assuming the flow as laminar, the head loss is given by

$$h_f = \frac{32\mu vL}{\gamma d^2}$$

$$\text{or, } 6 = \frac{32\times 0.01\times v\times 5}{8.64\times 1000\times (0.006)^2}$$

$$\text{or, } v = 1.17\, m/s$$

$$\text{Reynolds number, } N_{Re} = \frac{\rho vd}{\mu}$$

$$\text{or, } N_{Re} = \frac{8.64\times 1000\times 1.17\times 0.006}{9.81\times 0.01}$$

$$\text{or, } N_{Re} = 618.18$$

Since $N_{Re} \langle 2000$, the flow is laminar.

Flow rate, $Q = Av$

$$\text{or, } Q = \frac{\pi}{4}\times d^2\times v$$

$$\text{or, } Q = \frac{\pi}{4}\times (0.006)^2\times 1.17\times 1000$$

$$\text{or, } Q = 0.033\ \ l/s \qquad \textbf{\textit{Answer}}$$

Problem:

A certain liquid of specific gravity 0.85 is pumped through a horizontal pipe of 1 km long. The diameter of pipe is 125 mm and flow rate of liquid is 16 l/s. The pump absorbs 8 kW and its efficiency is 70%. Find the **dynamic viscosity of oil and type of flow**.

Given Data:

Diameter of pipe, d = 125 mm, Length of pipe, L = 1 km,
Specific gravity of oil, S = 0.85, Flow rate of oil, Q = 16 l/s = 0.016 m^3/s
Power absorbed by pump, P = 8 kW, Pump efficiency, η = 70%.

Solution:

Let us consider that the flow is laminar.

Specific weight of oil, $\gamma = 0.85 \times 9.81 = 8.34\, kN/m^3$

Average velocity of flow, $v = \dfrac{Q}{A} = \dfrac{0.016}{\dfrac{\pi}{4} \times (0.125)^2} = 1.3\, m/s$

Let h_f = head loss due to friction

Now for the pump, $\eta \times P = Q\gamma h_f$

$$\text{or, } 0.70 \times 8 = 0.016 \times 8.34 \times h_f$$

$$\text{or, } h_f = 41.97\,m$$

For laminar flow head loss due to friction is given b,

$$h_f = \frac{32\mu vL}{\gamma d^2}$$

$$\text{or, } 41.97 = \frac{32 \times \mu \times 1.3 \times 1}{8.34 \times (0.125)^2}$$

$$\text{or, } \mu = 0.13\,Ns/m^2 \quad \textit{Answer}$$

Now Reynolds number is given by,

$$N_{Re} = \frac{\rho vd}{\mu}$$

$$\text{or, } N_{Re} = \frac{0.25 \times 1000 \times 1.3 \times 0.125}{0.13}$$

$$\text{or, } N_{Re} = 1062.5$$

Since the Reynolds number is less than 2000, the flow is laminar.

Frictional Losses The drop in pressure is due to frictional resistance.

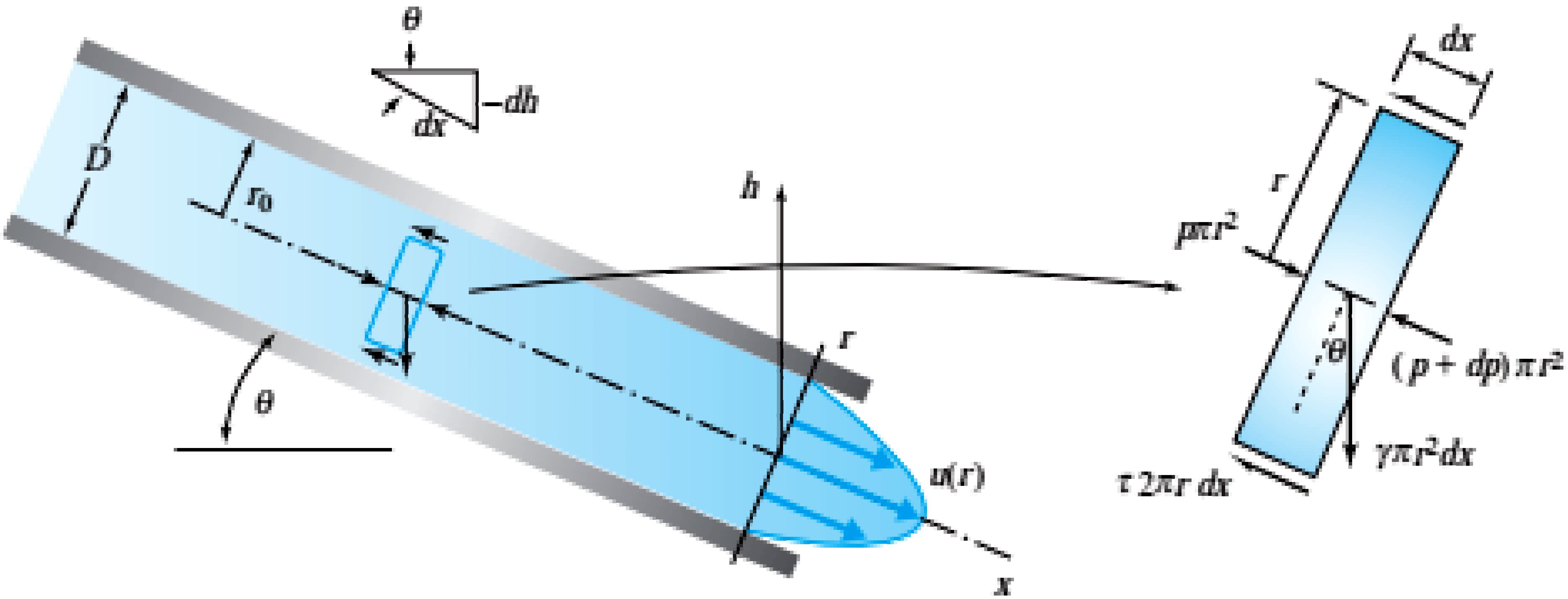


Fig: Developed flow in a circular pipe.

$$\tau = -\frac{r}{2}\frac{d}{dx}(p + \gamma h)$$

Force balance in the *x*-direction yields

$$p\pi r^2 - (p + dp)\pi r^2 - \tau 2\pi r\, dx + \gamma \pi r^2\, dx \sin\theta = 0$$

$$p_1 A - p_2 A + \gamma A z_1 - \gamma A z_2 = \tau_{wall} P L$$

$$(p_1 - p_2)A + \gamma A (z_1 - z_2) = \tau_{wall} P L$$

$$\gamma\left[\frac{\Delta p}{\gamma} + \Delta Z\right] = \frac{\tau_{wall} P L}{A}$$

Where P = Perimeter = πD and
A = *x*-sectional area = πR^2

$$\frac{P_1 - P_2}{\gamma} = \frac{\Delta p}{\gamma} + \Delta z = h_f = \textit{head loss due to friction}$$

$$\boxed{\therefore \quad \frac{\left[\frac{\Delta p}{\gamma} + \Delta z\right]}{L} = \frac{h_f}{L} = \frac{\tau_{wall} P}{A\gamma}}$$

Where hydraulic radius = A/P

$= \pi R^2/2\pi R = R/2 = D/4$

The total frictional (drag) force on the pipe due to laminar flow *u(r)* can be calculated from the wall shear stress τ_{max} that acts over the peripheral area $2\pi Rl$ for pipe length l.

$$F_{skin\ friction} = \tau_{max} A_s = C_f (\rho V^2/2) A_s$$

Where A_s is the total surface area that is in contact with the fluid and C_f is the skin friction coefficient (also called ***Fanning friction factor*** *named after* ***John Thomas Fanning***) is a **dimensionless number** used as a local parameter in continuum mechanics calculations. It is ***defined as the ratio between the local shear stress and the local flow kinetic energy density.***

$$\frac{h_f}{L} = \frac{2\tau_{wall}}{R\gamma} = \frac{2C_f\left(\rho V^2/_2\right)}{R\rho g}$$

Or,

$$h_f = C_f \frac{L}{R} \frac{V^2}{g}$$

From **the laminar pipe flow**, the shearing stress is at wall

$$\tau_w = - R/2\ [(p+\gamma h)/L] = R(P_1 - P_2)/2L$$

$$\boxed{\Delta p = 32\mu l\bar{u}/d^2.}$$

$$C_f = \tau_{max} / (\rho V^2/2)$$
$$= [R/2(\Delta p/L)] / (\rho V^2/2)$$
$$= [R/2\ (32\mu V/d^2)] / (\rho V^2/2)$$
$$= 16\mu /\rho V d \quad = 16/(\rho V d/\mu) \quad = 16/Re_d$$

$$h_f = C_f \frac{L}{R}\frac{V^2}{g}$$

Or,

$$\boxed{h_f = \left(\frac{16}{Re_D}\right)\frac{L}{R}\frac{V^2}{g}}$$

- ***Fanning friction factor is one-fourth of the Darcy friction factor, f.***

$$C_f = f/4 \qquad \text{so, } f = 4C_f$$

$$\boxed{h_f = C_f \frac{L}{R}\frac{V^2}{g} = \frac{f}{4}\frac{LV^2}{Rg} = f\frac{L}{D}\frac{V^2}{2g}}$$

This equation is called the **Darcy-Weisbach equation**. The head loss h_f represents ***the additional height that the fluid needs to be raised by a pump in order to overcome the frictional losses in the pipe***. The head loss is caused by viscosity, and it is directly related to the wall shear stress.

For **the laminar pipe flow**, ***$f = 4C_f = 64/Re_D$***

$$\boxed{h_f = f\frac{L}{D}\frac{V^2}{2g} = \left(\frac{64}{Re_D}\right)\frac{L}{R}\frac{V^2}{g}}$$